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FINITE GROUPS WITH A STANDARD COMPONENT WHOSE CENTRALIZER HAS CYCLIC SYLOW 2-SUBGROUPS

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ABSTRACT. Let G be a finite group with $O(G) = 1$, A a standard component of G and X the normal closure of A in G. Furthermore, assume that $C(A)$ has cyclic Sylow 2-subgroups. Then conditions are given on A which imply that either $X = A$ or $C(A)$ has Sylow 2-subgroups of order 2. These results are then applied to the cases where A is isomorphic to $\cdot 0$ or $\hat{R}u$, the proper 2-fold covering of the Rudvalis group.

1. **Introduction.** The combined work of Aschbacher and Seitz [2], [6] reduces the problem of classifying finite groups with a standard component A of known type to the case where $C(A)$ has cyclic Sylow 2-subgroups. The object of this paper is to show that in many cases, the problem may be further reduced to the case where $C(A)$ has Sylow 2-subgroups of order 2. Our main results are proved in §2 and are then applied in §§3 and 4 to the cases where A is isomorphic to $\hat{R}u$ and $\cdot 0$, respectively, with $\hat{R}u$ being the full covering group of the Rudvallis group.

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2. **Main results.**

THEOREM 1. *Let G be a finite group with $O(G) = 1$, A a standard component of G and $X = \langle A^G \rangle$. Let $K = C(A)$ and $R \in \mathrm{Syl}_2(K)$ with $R \cong Z_{2^n}$. Then one of the following occurs.*

(i) $X = A$.

(ii) $m(A) = 1$ *and X is isomorphic to* $L_3(q)$, $q \equiv -1 \pmod 4$, $q \neq 3$, *or* $U_3(1)$, $q \equiv 1 \pmod 4$.

(iii) $R^g \leqslant N(A)$ *for some* $g \in G - N(A)$.

PROOF. Assume that $X \neq A$ and let $g \in G - N(A)$ so that $Q = K^g \cap N(A)$ has Sylow 2 subgroup T of maximal order. If $|T| = |R|$, then (iii) holds. Hence assume that $|T| < |R|$. Since K is tightly embedded in G and 2-nilpotent by Burnside's transfer theorem, $N(R_1)$ covers $N(A)/O(N(A))$ for $1 \neq R_1 \leqslant R$. Thus if $|R_1| > |T|$ and $R_1^h \leqslant N(R_1)$ for some $h \in G$, then $h \in N(A)$, hence $R_1^h \leqslant N_K(R_1) = RN_{O(K)}(R_1)$ which forces $R_1^h = R_1$ and shows that R_1 is weakly closed in $N(R_1)$ with respect to G. This implies that

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$|T| > 1$, for otherwise $\Omega_1(R) \leqslant Z(G)$ by the Z^* theorem [10] and this contradicts $X \neq A$. Furthermore, we must then have $|R| > 2$ in which case X is simple and $G \leqslant \mathrm{Aut}(X)$ by Lemma 2.5 [4]. Let $\langle z \rangle = \Omega_1(R)$. If $|T| = 2$ and R_1 is the unique Z_4 subgroup of R, then by [5, 2.5], $\langle z^G \rangle$ has dihedral or semidihedral Sylow 2-subgroups. Since $\langle z^G \rangle \geqslant X$ and $[z, A] = 1$, it then follows that $z \in A$. Therefore A has 2 rank 1 and $\langle z^G \rangle = X$. By the results of [1], X must be isomorphic to $L_3(q)$, $q \equiv -1 \pmod 4$, $q \neq 3$ or $U_3(q)$, $q \equiv 1 \pmod 4$, hence (ii) holds.

We may now assume that $|T| > 2$ and, in addition, that T normalizes R. By [3, Theorem 2], $C_R(T) = N_R(T) \cong T$, hence our assumption that $|T| < |R|$ gives $C_R(T) < R$. Set $U = C_R(T)$ and let $U \leqslant V \leqslant R$ with $[V : U] = 2$. Now $UT \lhd VT$ and $N_{VT}(T) = N_{VT}(\Omega_1(T))$, so for $v \in V - U$, $UT = T \times T^v$. This implies that v inverts $[v, T]$ and since $V \lhd VT$, v centralizes $[v, T]$ as well. But $[v, T] \cong T$ and a contradiction is established.

The main result of this paper concerns quasisimple groups which satisfy

HYPOTHESIS I. A is a quasisimple group such that $Z(A)$ has a cyclic Sylow 2-subgroup. Let $\bar{A} = A/Z(A)$, $\bar{A} \leqslant \bar{L} \leqslant \mathrm{Aut}(\bar{A})$ and $\bar{t}$ an involution of $\bar{L}$. Then $C_{\bar{L}}(\bar{t})/O(C_{\bar{L}}(\bar{t}))$ has no Z_4 or $Z_2 \times Z_4$ normal subgroup.

THEOREM 2. *Let G be a finite group with $O(G) = 1$, A a standard component of G satisfying Hypothesis* I *and $X = \langle A^G \rangle$. Assume that $K = C(A)$ has a cyclic Sylow 2-subgroup. Then either $X = A$ or $|K|_2 = 2$.*

PROOF. Assume that $X \neq A$ and K has a Sylow 2-subgroup $R \cong Z_{2^n}$, $n > 1$. If $|R^g \cap N(A)| < |R|$ for all $g \in G - N(A)$, then by Theorem 1(ii), $A \cong SL_2(q)$, q odd, $q \neq 3$. But then A does not satisfy Hypothesis I, hence by Theorem 1(iii), there exists $g \in G - N(A)$ such that $T = R^g$ is a Sylow 2-subgroup of $Q = K^g \cap N(A)$. It is easy to see that Q is tightly embedded in $L = QA$. In fact, if $Q \cap Q^a$ has even order for $a \in A$, then so does $K^g \cap K^{ga}$, hence $K^g = K^{ga}$. This in turn forces $Q^a = (K^g \cap N(A))^a = K^g \cap N(A) = Q$ as required. Let $\bar{L} = L/L \cap K$ and $\langle t \rangle = \Omega_1(T)$. Then $\bar{A} \leqslant \bar{L} \leqslant \mathrm{Aut}(\bar{A})$ and $\bar{t}$ is an involution of $\bar{L}$.

Since Q is tightly embedded in L, $C_L(t)$ normalizes Q and this implies that $\overline{C_L(t)}$ normalizes $\bar{Q} = O(\bar{Q})\bar{T}$. Let $\bar{C} = C_{\bar{L}}(\bar{t})$. We claim that $\bar{C} = \overline{C_L(t)}$ except when $L \cap K$ has an involution z with $z \sim zt$ in L in which case $[\bar{C}: \overline{C_L(t)}] = 2$. In order to prove this it suffices to show that t centralizes a Sylow 2-subgroup R_0 of $L \cap K$. Without loss, we may assume that $R_0 \leqslant R$ and T normalizes R. But then $N_R(T) \cong T$ by [3, Theorem 2] whereupon $[R, T] = 1$. Let T_1 be the unique Z_4 subgroup of T. Observe that $[\bar{C}: \overline{C_L(t)}] \leqslant 2$ implies that $O(\bar{C}) = O(\overline{C_L(t)})$. Since $\bar{Q} \cap \overline{C_L(t)}$ is a 2-nilpotent normal subgroup of $\overline{C_L(t)}$ with Sylow 2-subgroup $\bar{T}$, it then follows that $\bar{T}O(\bar{C}) \lhd \overline{C_L(t)}$. Hence $\bar{T}_1 O(\bar{C}) \lhd \overline{C_L(t)}$. By hypothesis, $\bar{C}/O(\bar{C})$ has no Z_4 or $Z_2 \times Z_4$ normal subgroup, therefore we must have $\bar{C} = \langle \overline{C_L(t)}, \bar{\mu} \rangle$ where

$$\langle \bar{T}_1, \bar{T}_1^{\bar{\mu}} \rangle O(\bar{C}) \lhd \bar{C}.$$

Assuming that $\bar{T}_1^{\bar{\mu}}$ normalizes $\bar{T}_1$ and again invoking our hypothesis in conjunction with the fact that $\Omega_1(\bar{T}_1) = \Omega_1(\bar{T}_1^{\bar{\mu}}) = \langle \bar{t} \rangle$, we have $\langle \bar{T}_1, \bar{T}_1^{\bar{\mu}} \rangle \cong Q_8$. A contradiction may now be established by noting that if $\bar{T}_0$ is the Z_4 subgroup of $\langle \bar{T}_1, \bar{T}_1^{\bar{\mu}} \rangle$ not equal to $\bar{T}_1$ or $\bar{T}_1^{\bar{\mu}}$, then $\bar{T}_0 O(\bar{C}) \lhd \bar{C}$.

Theorem 2 extends [9, Proposition 5.2]. Many, but not all, of the sporadic groups satisfy Hypothesis I and, therefore, Theorem 2 should be useful in these standard component problems.

Suppose now that G and A satisfy the hypotheses of Theorem 2 except that A does not satisfy Hypothesis I. Then the conclusion of Theorem 2 no longer holds. In fact, counterexamples exist when G is isomorphic to Aut(F_5) [12] with A isomorphic to the proper 2-fold covering of the Higman-Sims group HS and G is isomorphic to the O'Nan-Sims group [13] with A isomorphic to a perfect central extension of Z_4 by $L_3(4)$. Note that HS has an involution whose centralizer has a normal Z_4 subgroup whereas $L_3(4)$ has an involution whose centralizer has a normal $Z_2 \times Z_4$ subgroup. At any rate, Theorem 1 may be helpful in these problems since it asserts that if $X \neq A$, then there exists $g \in G - N(A)$ such that $Q = K^g \cap N(A)$ has Sylow 2-subgroup R^g and Q is tightly embedded in QA with R^g acting faithfully on A. One should then be able to use the structure of A to determine $|R|$.

3. **The case** $A \cong \hat{Ru}$. Let Ru be the simple group of order $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ whose existence was first proposed by Rudvalis and later confirmed by Conway and Wales [7]. The multiplier of Ru is known to have order 2 and the outer automorphism group is trivial (see [11]). The character tables of both Ru and $\hat{Ru}$ have been computed by Frame and Rudvalis. The elements of 2 power order of $\hat{Ru}$ are listed in the following table:

TABLE I

$\hat{Ru}$ Class	Centralizer Order	Squares
± 1	$2^{15} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	1
$\pm 2A$	$2^{15} \cdot 3 \cdot 5$	1
$4A$	2^{10}	$-2A$
$\pm 4B$	$2^{10} \cdot 3 \cdot 5$	$2A$
$4C$	2^9	$2A$
$\pm 4D$	$2^9 \cdot 3 \cdot 5$	$-2A$
$4E$	$2^8 \cdot 5 \cdot 7 \cdot 13$	-1
$\pm 8A$	2^7	$4A$
$8B$	2^5	$4C$
$\pm 8C$	$2^6 \cdot 3 \cdot 5$	$-4B$
$\pm 16A$	2^5	$8A$
$\pm 16B$	2^5	$-8A$

Here nA represents a class of elements of order n, $\langle -1 \rangle$ is the center and $\pm nA$ denotes a double class of elements.

LEMMA 3.1. *Let* $A \cong \hat{Ru}$. *Then* A *satisfies Hypothesis* I.

PROOF. Let $\bar{A} = A/Z(A)$ and $\bar{t}$ be an involution of $\bar{A}$. It is clear from Table I that $\bar{A}$ has 2 classes of involutions, denoted by $2A$ and $2B$. Furthermore it is known that if $\bar{t} \in 2A$ then $C_{\bar{A}}(\bar{t})$ is 2-constrained of order $2^{14} \cdot 3 \cdot 5$ with $C_{\bar{A}}(\bar{t})/O_2(C_{\bar{A}}(\bar{t})) \cong S_5$ whereas if $\bar{t} \in 2B$, then $C_{\bar{A}}(\bar{t}) \cong E_4 \times S_z(8)$ (see [11]). Moreover, if $\bar{s}$ is an element of order 4, then by Table I, $|C_{\bar{A}}(\bar{s})|_2 \leqslant 2^{10}$. It now follows that in either case, $C_{\bar{A}}(\bar{t})$ has no Z_4 or $Z_2 \times Z_4$ normal subgroup and $O(C_{\bar{A}}(\bar{t})) = 1$. Thus A satisfies Hypothesis I as required.

THEOREM 3. *Let G be a finite group with $O(G) = 1$, A a standard component of G isomorphic to $\hat{R}u$ and $X = \langle A^G \rangle$. Furthermore assume that $K = C(A)$ has cyclic Sylow 2-subgroups. Then $X = A$.*

PROOF. Assume by way of a contradiction that $X \neq A$. It then follows from Lemma 3.1 and Theorem 2 that $\langle z \rangle = Z(A) \in \mathrm{Syl}_2(K)$. Setting $L = C(z)$, we have $L = O(L) \times A$. It is clear from Table I that all involutions of $A - Z(A)$ are 4th powers whereas z is not. Therefore z is isolated in A, hence in L as well and we may conclude from the Z^* theorem [10] that $z \in Z(G)$. This is incompatible with $X \neq A$ and the proof is completed.

4. **The case** $A \cong \cdot 0$. We begin by enumerating well-known properties of $\cdot 0$ (see [6]). $\cdot 0$ is a perfect group with center of order 2 and central quotient group isomorphic to $\cdot 1$. The outer automorphism group of $\cdot 1$ is trivial and $\cdot 0$ is a representation group for $\cdot 1$. Moreover $\cdot 1$ has 3 classes of involutions, denoted by (2_1), (2_2) and (2_3). If $r_i \in (2_i)$ and $L_i = C_{\cdot 1}(r_i)$, then L_1 is an extension of $Q_8 * Q_8 * Q_8 * Q_8$ by $\Omega_8^+(2)$, L_2 is an extension of $E_{2^{11}}$ by $\mathrm{Aut}(M_{12})$ and L_3 is isomorphic to $E_4 \times G_2(4)$ extended by an involution which acts as the field automorphism on $G_2(4)$ and together with $O_2(L_3)$ generates a D_8 subgroup. It is now evident that the following holds.

LEMMA 4.1. *$\cdot 1$ satisfies Hypothesis* I.

We can now prove the main result of this section.

THEOREM 4. *Let G be a finite group with $O(G) = 1$, A a standard component of G isomorphic to $\cdot 0$ and $X = \langle A^G \rangle$. Assume that $K = C(A)$ has cyclic Sylow 2-subgroups. Then $X = A$.*

PROOF. Assume that $X \neq A$. It then follows from Lemma 4.1 and Theorem 2 that $\langle z \rangle = Z(A) \in \mathrm{Syl}_2(K)$. Thus $C(z) = A^* \times A$ where $A^* = O(C(z))$. Now finite groups in which the centralizer of an involution is isomorphic to $\cdot 0$ have been classified [8]. In particular, such groups, modulo core, are isomorphic to $\cdot 0$. Moreover, the proof of [8] may be easily adapted to our present case and we conclude that $z \in Z^*(G)$, contradicting $X \neq A$.

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