BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PARTIAL DIFFERENCE EQUATION

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Abstract. Let $\mathcal{G}$ be a partial difference operator with constant coefficients in $n$ independent (discrete) variables, and let $\mathcal{S}_\mathcal{G} = \{ f: \mathbb{Z}^n \to \mathbb{C}; \mathcal{G}f = 0 \}$. We introduce a certain class of binary operations $\mathcal{S}_\mathcal{G} \times \mathcal{S}_\mathcal{G} \to \mathcal{S}_\mathcal{G}$ generalizing a binary operation introduced by Duffin and Rohrer.

1. Introduction. Let $\mathbb{Z}^n$ be the $n$-dimensional lattice and consider a partial difference operator on $\mathbb{Z}^n$

$$\mathcal{G}f(m) = \sum_{|k| \leq N} C_k f(m + k),$$

where $m, k \in \mathbb{Z}^n$, $|k| = \sum_{i=1}^{N} |k_i|$, $k = (k_1, \ldots, k_n)$ and $N$ is an integer. In this note we shall characterize all products $\ast$ of the form

$$\sum_{r \in \mathbb{Z}^n, k \in \mathbb{Z}^n} d_{kr}^m f(r) g(k)$$

(only a finite number of terms on the right-hand side being nonzero) with the property that if $\mathcal{G}f = 0$ and $\mathcal{G}g = 0$ then $\mathcal{G}(f \ast g) = 0$. The product of Duffin and Rohrer [1] falls in this category. The basic idea is to associate with every discrete function $f: \mathbb{Z}^n \to \mathbb{C}$ a linear functional $\mathcal{T}_f$ on the algebra $\mathcal{A}_n$ generated by the indeterminates $\{z_1, z_{-1}^1, \ldots, z_n, z_{-1}^n\}$, given by

$$\mathcal{T}_f(z_1^{k_1}, \ldots, z_n^{k_n}) = f(k_1, \ldots, k_n)$$

for every $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and extended by linearity. Conversely, (1.2) associates a discrete function $f: \mathbb{Z}^n \to \mathbb{C}$ to every such linear functional.

2. Binary operations on the set of solutions of $\mathcal{G}u = 0$. DEFINITION 2.1. Any operation $(f, g) \to f \ast g$ which maps pairs of functions on $\mathbb{Z}^n$ to another function on $\mathbb{Z}^n$ and is of the form (1.1) will be termed a Duffin product.

LEMMA 2.2. Any Duffin product induces a linear mapping $\mathcal{F}: \mathcal{A}_n \to \mathcal{A}_{2n}$ such that if $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_n)$,
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(2.1) \( T_f g(u(z)) = T_f T_g(\mathcal{S}u(z,t)) \)

where \( T_f T_g \) is the linear functional on \( \mathcal{A}_{2n} \) defined by

(2.2) \( T_f T_g(z^k r') = T_f(z^k)T_g(r') \)

and extended by linearity.

**Proof.** By (1.1)

\[
T_f g(z^m) = (f \ast g)(m) = \sum d_{kr} T_f(z^k T_g(r') = T_f T_g(\sum d_{kr} z^k r').
\]

Define \( \mathcal{S}(z^m) = \sum d_{kr} z^k r' \) and extend by linearity. Obviously (2.1) defines a Duffin product for each such mapping.

**Lemma 2.3.** Let \( \mathcal{B} \) be a partial difference operator with constant coefficients \( \mathcal{B} f(m) = \sum C_k f(m + k) \), and let \( P(z) \in \mathcal{A}_n \) be its symbol, \( P(z) = \sum C_k z^k \).

Then \( \mathcal{B} f \equiv 0 \) iff \( T_f \) annihilates the principal ideal \( P(z)\mathcal{A}_n = \{P(z)u(z); u(z) \in \mathcal{A}_n\} \).

**Proof.** The statement is self-evident from the identity

\[
T_f (P(z)z^m) = T_f (\sum C_k z^{m+k}) = \sum C_k f(m + k).
\]

Now we are in a position to prove our central result.

**Theorem.** A Duffin product induced by the mapping \( \mathcal{B}: \mathcal{A}_n \to \mathcal{A}_{2n} \), given in Lemma 2.2, maps pairs of solutions of \( \mathcal{B}u = 0 \) into another solution if \( \mathcal{B}(P(z)\mathcal{A}_n) \) is contained in the ideal generated by \( \{P(z), P(t)\} \), i.e., if for every \( u(z) \in \mathcal{A}_m \) we can find \( a(z,t), b(z,t) \in \mathcal{A}_{2n} \) such that

\[
\mathcal{B}(P(z)u(z)) = a(z,t)P(z) + b(z,t)P(t).
\]

**Proof.** \( \mathcal{B} f \equiv 0 \) if \( T_f g(P(z)\mathcal{A}_n) = 0 \). Now

\[
T_f g(P(z)u(z)) = T_f T_g(\mathcal{B}P(z)u(z)) = T_f T_g(a(z,t)P(z) + b(z,t)P(t)) = 0.
\]

3. **Applications.** The theorem makes very easy the verification that a given Duffin product preserves the property of being a solution of a given partial difference equation with constant coefficients. This will be illustrated by the following two examples.

(a) Duffin and Duris [2] introduced three kinds of 'convolution products' for solutions of the discrete Cauchy-Riemann equation.

(3.1) \( f(m,n) + if(m + 1,n) = f(m + 1,n + 1) - if(m,n + 1) = 0. \)

They denoted them by \( f \ast g, f \ast' g \) and \( f \ast'' g \). An easy calculation, which is not reproduced here in order to save space, shows that the corresponding mappings \( \mathcal{B}, \mathcal{B}', \mathcal{B}'' : \mathcal{A}_2 \to \mathcal{A}_4 \) are (make the notational transformation \( z = (z_1, z_2) = (z, w), t = (t_1, t_2) = (t, s) \))
\[ \mathcal{F}: u(z, w) \rightarrow (1 + i)(1 + z)\frac{u(z, w) - u(t, w)}{z - t} + i(1 + s)(1 + w)\frac{u(t, w) - u(t, s)}{w - s}, \]

\[ \mathcal{F}': u(z, w) \rightarrow (1 + z)(1 - t)\frac{u(z, w) - u(t, w)}{z - t} + i(1 - s)(1 + w)\frac{u(t, w) - u(t, s)}{w - s}, \]

\[ \mathcal{F}'': u(z, w) \rightarrow (1 - z)(1 - t)\frac{u(z, w) - u(t, w)}{z - t} + i(1 - s)(1 - w)\frac{u(t, w) - u(t, s)}{w - s}. \]

From these formulas we deduce easily that the corresponding convolution products indeed preserve discrete-analyticity (i.e., the property of being a solution of (3.1)). They can also be used to advantage in giving short proofs of the commutativity and associativity of these products.

(b) For a general partial difference equation with constant coefficients \( \mathcal{P}u = 0 \), in \( \mathbb{Z}^2 \), Duffin and Rohrer [1] introduced a 'product' which can be shown, by a straightforward but a little lengthy calculation, to be induced by

\[ \mathcal{F}(u(z, w)) = ts \left\{ \frac{u(t, s) - u(t, w)}{s - w} \left[ \frac{P(z, w) - P(t, w)}{z - t} \right] - \frac{u(z, w) - u(t, w)}{z - t} \left[ \frac{P(t, s) - P(t, w)}{s - w} \right] \right\} \]

\[ = \frac{ts}{(s - w)(z - t)} [u(t, s)[P(z, w) - P(t, w)] - u(t, w)[P(z, w) - P(t, s)] - u(z, w)[P(t, s) - P(t, w)]], \]

where \( P(z, w) \) is the symbol of \( \mathcal{P} \). \( \mathcal{F} \) is seen to satisfy the hypothesis of the Theorem, thus furnishing a short proof to the fact that if \( \mathcal{P}f = 0 \) and \( \mathcal{P}g = 0 \), then \( \mathcal{P}(f \circ g) = 0 \) (see Duffin and Rohrer [1, pp. 691–693] for the original proof).

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**References**


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