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BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PARTIAL DIFFERENCE EQUATION

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ABSTRACT. Let $\mathcal{P}$ be a partial difference operator with constant coefficients in n independent (discrete) variables, and let $\mathbb{S}_{\mathcal{P}} = \{f: Z^n \to \mathbf{C}; \mathcal{P}f = 0\}$. We introduce a certain class of binary operations $\mathbb{S}_{\mathcal{P}} \times \mathbb{S}_{\mathcal{P}} \to \mathbb{S}_{\mathcal{P}}$ generalizing a binary operation introduced by Duffin and Rohrer.

1. **Introduction.** Let Z^n be the n-dimensional lattice and consider a partial difference operator on Z^n

$$\mathcal{P}f(m) = \sum_{|k| \leqslant N} C_k f(m+k),$$

where $m, k \in Z^n$, $|k| = \sum_{i=1}^N |k_i|$, $k = (k_1, \ldots, k_n)$ and N is an integer. In this note we shall characterize all products $*$ of the form

$$(f * g)(m) = \sum_{r \in Z^n; k \in Z^n} d_{kr}^m f(r) g(k) \tag{1.1}$$

(only a finite number of terms on the right-hand side being nonzero) with the property that if $\mathcal{P}f \equiv 0$ and $\mathcal{P}g \equiv 0$ then $\mathcal{P}(f * g) \equiv 0$. The product of Duffin and Rohrer [1] falls in this category. The basic idea is to associate with every discrete function f: $Z^n \to \mathbf{C}$ a linear functional T_f on the algebra $\mathcal{Q}_n$ generated by the indeterminates $\{z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}\}$, given by

$$T_f(z_1^{k_1}, \ldots, z_n^{k_n}) = f(k_1, \ldots, k_n) \tag{1.2}$$

for every $(k_1, \ldots, k_n) \in Z^n$ and extended by linearity. Conversely, (1.2) associates a discrete function f: $Z^n \to \mathbf{C}$ to every such linear functional.

2. **Binary operations on the set of solutions of $\mathcal{P}u \equiv 0$.**

DEFINITION 2.1. Any operation $(f, g) \to f * g$ which maps pairs of functions on Z^n to another function on Z^n and is of the form (1.1) will be termed a *Duffin product*.

LEMMA 2.2. *Any Duffin product induces a linear mapping* $\mathcal{F}$: $\mathcal{Q}_n \to \mathcal{Q}_{2n}$ *such that if* $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_n)$,

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$$T_{f*g}(u(z)) = T_f T_g(\mathcal{F}u(z,t)) \tag{2.1}$$

where $T_f T_g$ is the linear functional on $\mathcal{Q}_{2n}$ defined by

$$T_f T_g(z^k t^r) = T_f(z^k) T_g(t^r) \tag{2.2}$$

and extended by linearity.

PROOF. By (1.1)

$$T_{f*g}(z^m) = (f * g)(m) = \sum d_{kr}^m T_f(z^k) T_g(t^r) = T_f T_g(\sum d_{kr}^m z^k t^r).$$

Define $\mathcal{F}(z^m) = \sum d_{kr}^m z^k t^r$ and extend by linearity. Obviously (2.1) defines a Duffin product for each such mapping.

LEMMA 2.3. *Let $\mathcal{P}$ be a partial difference operator with constant coefficients $\mathcal{P}f(m) = \sum C_k f(m+k)$, and let $P(z) \in \mathcal{Q}_n$ be its symbol, $P(z) = \sum C_k z^k$. Then $\mathcal{P}f \equiv 0$ iff T_f annihilates the principal ideal $P(z)\mathcal{Q}_n = \{P(z)u(z); u(z) \in \mathcal{Q}_n\}$.*

PROOF. The statement is self-evident from the identity

$$T_f(P(z)z^m) = T_f(\sum C_k z^{m+k}) = \sum C_k f(m+k).$$

Now we are in a position to prove our central result.

THEOREM. *A Duffin product induced by the mapping $\mathcal{F}: \mathcal{Q}_n \to \mathcal{Q}_{2n}$, given in Lemma 2.2, maps pairs of solutions of $\mathcal{P}u \equiv 0$ into another solution if $\mathcal{F}(P(z)\mathcal{Q}_n)$ is contained in the ideal generated by $\{P(z), P(t)\}$, i.e., if for every $u(z) \in \mathcal{Q}_m$ we can find $a(z,t)$, $b(z,t) \in \mathcal{Q}_{2n}$ such that*

$$\mathcal{F}(P(z)u(z)) = a(z,t)P(z) + b(z,t)P(t).$$

PROOF. $\mathcal{P}(f * g) \equiv 0$ if $T_{f*g}(P(z)\mathcal{Q}_n) = 0$. Now

$$T_{f*g}(P(z)u(z)) = T_f T_g(\mathcal{F}P(z)u(z)) = T_f T_g(a(z,t)P(z) + b(z,t)P(t)) = 0.$$

3. **Applications.** The theorem makes very easy the verification that a given Duffin product preserves the property of being a solution of a given partial difference equation with constant coefficients. This will be illustrated by the following two examples.

(a) Duffin and Duris [2] introduced three kinds of 'convolution products' for solutions of the discrete Cauchy-Riemann equation.

$$f(m,n) + if(m+1,n) - f(m+1,n+1) - if(m,n+1) \equiv 0. \tag{3.1}$$

They denoted them by $f * g$, $f *' g$ and $f *'' g$. An easy calculation, which is not reproduced here in order to save space, shows that the corresponding mappings $\mathcal{F}, \mathcal{F}', \mathcal{F}'': \mathcal{Q}_2 \to \mathcal{Q}_4$ are (make the notational transformation $z = (z_1, z_2) = (z, w)$, $t = (t_1, t_2) = (t, s)$)

$$\mathcal{F}\colon u(z,w) \to (1+t)(1+z)\frac{u(z,w)-u(t,w)}{z-t} + i(1+s)(1+w)\frac{u(t,w)-u(t,s)}{w-s},$$

$$\mathcal{F}'\colon u(z,w) \to (1+z)(1-t)\frac{u(z,w)-u(t,w)}{z-t} + i(1-s)(1+w)\frac{u(t,w)-u(t,s)}{w-s},$$

$$\mathcal{F}''\colon u(z,w) \to (1-z)(1-t)\frac{u(z,w)-u(t,w)}{z-t} + i(1-s)(1-w)\frac{u(t,w)-u(t,s)}{w-s}.$$

From these formulas we deduce easily that the corresponding convolution products indeed preserve discrete-analyticity (i.e., the property of being a solution of (3.1)). They can also be used to advantage in giving short proofs of the commutativity and associativity of these products.

(b) For a general partial difference equation with constant coefficients $\mathcal{P}u \equiv 0$, in Z^2, Duffin and Rohrer [1] introduced a ' product' which can be shown, by a straightforward but a little lengthy calculation, to be induced by

$$\begin{aligned}\mathcal{F}(u(z,w)) &= ts\left\{\frac{u(t,s)-u(t,w)}{s-w}\left[\frac{P(z,w)-P(t,w)}{z-t}\right] - \frac{u(z,w)-u(t,w)}{z-t}\left[\frac{P(t,s)-P(t,w)}{s-w}\right]\right\} \\ &= \frac{ts}{(s-w)(z-t)}[u(t,s)[P(z,w)-P(t,w)] \\ &\qquad -u(t,w)[P(z,w)-P(t,s)] \\ &\qquad - u(z,w)[P(t,s)-P(t,w)]],\end{aligned}$$

where $P(z,w)$ is the symbol of $\mathcal{P}$. $\mathcal{F}$ is seen to satisfy the hypothesis of the Theorem, thus furnishing a short proof to the fact that if $\mathcal{P}f \equiv 0$ and $\mathcal{P}g \equiv 0$, then $\mathcal{P}(f * g) \equiv 0$ (see Duffin and Rohrer [1, pp. 691–693] for the original proof).

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REFERENCES

1. R. J. Duffin and Joan Rohrer Hundhausen, *A convolution product for the solutions of partial difference equations*, Duke Math. J. **35** (1968), 683–698. MR **39** #1831.

2. R. J. Duffin and C. S. Duris, *A convolution product for discrete function theory*, Duke Math. J. **31** (1964), 199–220. MR **29** #429.

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