BINARY OPERATIONS IN THE SET OF SOLUTIONS OF A PARTIAL DIFFERENCE EQUATION

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Abstract. Let \( \mathcal{Q} \) be a partial difference operator with constant coefficients in \( n \) independent (discrete) variables, and let \( \mathcal{S}_\mathcal{Q} = \{ f: \mathbb{Z}^n \to \mathbb{C}; \mathcal{Q}f = 0 \} \). We introduce a certain class of binary operations \( \mathcal{S}_\mathcal{Q} \times \mathcal{S}_\mathcal{Q} \to \mathcal{S}_\mathcal{Q} \) generalizing a binary operation introduced by Duffin and Rohrer.

1. Introduction. Let \( \mathbb{Z}^n \) be the \( n \)-dimensional lattice and consider a partial difference operator on \( \mathbb{Z}^n \)

\[
\mathcal{Q}f(m) = \sum_{|k| \leq N} C_k f(m + k),
\]

where \( m, k \in \mathbb{Z}^n, |k| = \sum_{i=1}^{N} |k_i|, k = (k_1, \ldots, k_n) \) and \( N \) is an integer. In this note we shall characterize all products * of the form

\[
(f \ast g)(m) = \sum_{r \in \mathbb{Z}^n, k \in \mathbb{Z}^n} d_{kr}^m f(r) g(k)
\]

(only a finite number of terms on the right-hand side being nonzero) with the property that if \( \mathcal{Q}f = 0 \) and \( \mathcal{Q}g = 0 \) then \( \mathcal{Q}(f \ast g) = 0 \). The product of Duffin and Rohrer [1] falls in this category. The basic idea is to associate with every discrete function \( f: \mathbb{Z}^n \to \mathbb{C} \) a linear functional \( T_f \) on the algebra \( \mathcal{S}_n \) generated by the indeterminates \( \{z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}\} \), given by

\[
T_f(z_{k_1}^{k_1}, \ldots, z_{k_n}^{k_n}) = f(k_1, \ldots, k_n)
\]

for every \( (k_1, \ldots, k_n) \in \mathbb{Z}^n \) and extended by linearity. Conversely, (1.2) associates a discrete function \( f: \mathbb{Z}^n \to \mathbb{C} \) to every such linear functional.

2. Binary operations on the set of solutions of \( \mathcal{Q}u = 0 \).

Definition 2.1. Any operation \((f,g) \to f \ast g\) which maps pairs of functions on \( \mathbb{Z}^n \) to another function on \( \mathbb{Z}^n \) and is of the form (1.1) will be termed a Duffin product.

Lemma 2.2. Any Duffin product induces a linear mapping \( \mathcal{F}: \mathcal{S}_n \to \mathcal{S}_{2n} \) such that if \( z = (z_1, \ldots, z_n), t = (t_1, \ldots, t_n) \),

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(2.1) \[ T_{f\circ g}(u(z)) = T_fT_g(\mathcal{S}u(z,t)) \]

where \( T_fT_g \) is the linear functional on \( \mathfrak{S}_{2n} \) defined by

(2.2) \[ T_fT_g(z^k t^r) = T_f(z^k)T_g(t^r) \]

and extended by linearity.

**Proof.** By (1.1)

\[ T_{f\circ g}(z^m) = (f \ast g)(m) = \sum d_{kr} T_f(z^k)T_g(t^r) = T_fT_g(\sum d_{kr} z^k t^r). \]

Define \( \mathcal{S}(z^m) = \sum d_{kr} z^k t^r \) and extend by linearity. Obviously (2.1) defines a Duffin product for each such mapping.

**Lemma 2.3.** Let \( \mathcal{P} \) be a partial difference operator with constant coefficients \( \mathcal{P} f(m) = \sum C_k f(m + k) \), and let \( P(z) \in \mathfrak{S}_n \) be its symbol, \( P(z) = \sum C_k z^k \). Then \( \mathcal{P} f = 0 \) iff \( T_f \) annihilates the principal ideal \( P(z)\mathfrak{S}_n = \{P(z)u(z); u(z) \in \mathfrak{S}_n\} \).

**Proof.** The statement is self-evident from the identity

\[ T_f(P(z)z^m) = T_f(\sum C_k z^{m+k}) = \sum C_k f(m + k). \]

Now we are in a position to prove our central result.

**Theorem.** A Duffin product induced by the mapping \( \mathcal{S}: \mathfrak{S}_n \rightarrow \mathfrak{S}_{2n} \), given in Lemma 2.2, maps pairs of solutions of \( \mathcal{P}u = 0 \) into another solution if \( \mathcal{S}(P(z)\mathfrak{S}_n) \) is contained in the ideal generated by \( \{P(z), P(t)\} \), i.e., if for every \( u(z) \in \mathfrak{S}_n \) we can find \( a(z,t), b(z,t) \in \mathfrak{S}_n \) such that

\[ \mathcal{S}(P(z)u(z)) = a(z,t)P(z) + b(z,t)P(t). \]

**Proof.** \( \mathcal{S}(f \ast g) = 0 \) if \( T_{f\circ g}(P(z)\mathfrak{S}_n) = 0 \). Now

\[ T_{f\circ g}(P(z)u(z)) = T_fT_g(\mathcal{S}P(z)u(z)) = T_fT_g(a(z,t)P(z) + b(z,t)P(t)) = 0. \]

3. **Applications.** The theorem makes very easy the verification that a given Duffin product preserves the property of being a solution of a given partial difference equation with constant coefficients. This will be illustrated by the following two examples.


(3.1) \[ f(m,n) + if(m + 1, n) - f(m + 1, n + 1) - if(m, n + 1) = 0. \]

They denoted them by \( f \ast g, f \ast' g \) and \( f \ast'' g \). An easy calculation, which is not reproduced here in order to save space, shows that the corresponding mappings \( \mathcal{S}, \mathcal{S}', \mathcal{S}'' : \mathfrak{S}_2 \rightarrow \mathfrak{S}_4 \) are (make the notational transformation \( z = (z_1, z_2) = (z, w), t = (t_1, t_2) = (t, s) \))
From these formulas we deduce easily that the corresponding convolution products indeed preserve discrete-analyticity (i.e., the property of being a solution of (3.1)). They can also be used to advantage in giving short proofs of the commutativity and associativity of these products.

(b) For a general partial difference equation with constant coefficients $\mathcal{P}u = 0$, in $\mathbb{Z}^2$, Duffin and Rohrer [1] introduced a 'product' which can be shown, by a straightforward but a little lengthy calculation, to be induced by

$$\mathcal{F}(u(z, w)) = \frac{ts}{(s - w)(z - t)} \left\{ u(t, s)P(z, w) - u(z, w)P(t, s) - u(z, w)[P(z, w) - P(t, s)] - u(t, w)[P(z, w) - P(t, s)] \right\},$$

where $P(z, w)$ is the symbol of $\mathcal{P}$. $\mathcal{F}$ is seen to satisfy the hypothesis of the Theorem, thus furnishing a short proof to the fact that if $\mathcal{P}f = 0$ and $\mathcal{P}g = 0$, then $\mathcal{P}(f \ast g) = 0$ (see Duffin and Rohrer [1, pp. 691–693] for the original proof).

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REFERENCES