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UNIQUE HAHN-BANACH EXTENSIONS AND SIMULTANEOUS EXTENSIONS

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ABSTRACT. In this note we mention certain connections existing between unique Hahn-Banach extensions and simultaneous extensions. We also describe an application of a continuous selection theorem to simultaneous extensions.

1. **Effects of unique Hahn-Banach extensions.** Let (m) be the Banach space of bounded sequences of real (or complex) numbers with the supremum norm and (c_0) the subspace of (m) consisting of sequences converging to zero. Then the following holds, which is ascribed to H. P. Rosenthal (cf. Banilower [1, Proposition 1.2]): If T is a linear operator from (m) into a normed linear space F with $\|T\| = 1$ such that T is an isometry on (c_0), then T is an isometry on (m). If, moreover, $F = (m)$ and T is the identity operator on (c_0), then T is the identity operator on (m). Our first objective is to describe principles underlying this result.

Let E be a normed linear space with dual E^*, L a subspace of E, T a linear operator from E into a normed linear space F with $\|T\| = 1$ such that T is an isometry on L, and $M = T(L)$, which is considered as a subspace of F. These notations are preserved through this section. Let $S^*(E,L)$ be the set of $x^* \in E^*$ with $\|x^*\| = 1$ such that $x^*|L = y^*|L$ implies $x^* = y^*$ if $\|y^*\| \leqslant 1$. If $x^* \in S^*(E,L)$, then $\|x^*|L\| = 1$ and x^* is the unique Hahn-Banach extension of $x^*|L$ to E. We denote by $E^*(L)$ the subspace of E^* generated by the $\sigma(E^*,E)$-closed convex hull of $S^*(E,L)$ and by $E^*(L)^{\perp}$ the orthogonal complement of $E^*(L)$ in the space E. By the Krein-Šmulian theorem (cf. Dunford [3, V.5.8]), $E^*(L)$ is $\sigma(E^*,E)$-closed. For a given subset D of E we say that an element $x \in E$ is orthogonal to D if $\|x + cx'\| \geqslant \|x\|$ for any $x' \in D$ and any scalar c. Consider the case in which D is a closed subspace. Since $D^{\perp}$, the orthocomplement of D in E^*, is regarded as the dual of the quotient space E/D, we have

$$\|x\| \geqslant \inf\{\|x + x'\|: x' \in D\} = \sup\{|\langle x, x^*\rangle|: x^* \in D^{\perp} \text{ and } \|x^*\| = 1\}.$$

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The equality sign prevails if and only if x is orthogonal to D. As for the isometric property of T we have the following

THEOREM 1. *$\|Tx\| = \|x\|$ for $x \in E$, if either $x \in L$ or x is orthogonal to the subspace $E^*(L)^{\perp}$.*

PROOF. Take any $x^* \in S^*(E,L)$ and set $v^* = x^* \circ (T|L)^{-1}$. Then v^* is a linear functional on the subspace $M = T(L)$ and $\|v^*\| = \|x^*|L\| = \|x^*\| = 1$, for T is an isometry on L. Let y^* be any Hahn-Banach extension of v^* to F. $y^* \circ T$ is then a Hahn-Banach extension of $x^*|L$ and therefore $x^* = y^* \circ T$. For any $x \in E$ we have $\|Tx\| \geqslant |\langle Tx, y^* \rangle| = |\langle x, y^* \circ T \rangle| = |\langle x, x^* \rangle|$ and so $\|Tx\| \geqslant \sup\{|\langle x, x^* \rangle|: x^* \in S^*(E,L)\}$. If x is orthogonal to $E^*(L)^{\perp}$, then the above remark with $D = E^*(L)^{\perp}$ shows that

$$\begin{aligned} \|x\| &= \sup\{|\langle x, x^* \rangle|: x^* \in E^*(L), \|x^*\| = 1\} \\ &= \sup\{|\langle x, x^* \rangle|: x^* \in S^*(E,L)\}. \end{aligned}$$

Hence $\|Tx\| \geqslant \|x\|$, which, together with $\|T\| = 1$, implies $\|Tx\| = \|x\|$. If $x \in L$, then $\|Tx\| = \|x\|$, for T is an isometry on L.

COROLLARY 2. *If $E^*(L) = E^*$, then T is an isometry on E.*

After Phelps [7] we say that a subspace L of E has property U in E if every bounded linear functional on L has a unique Hahn-Banach extension to E, i.e., $S^*(E,L)$ coincides with the set of $x^* \in E^*$ for which $\|x^*\| = \|x^*|L\| = 1$.

COROLLARY 3. *If L has property U in E and if $E \subseteq L^{**}$, then T is an isometry on E.*

We next consider another aspect of Rosenthal's result. For a subset $\mathfrak{L}$ of L^* let $H(E,\mathfrak{L})$ be the set of $x \in E$ such that all Hahn-Banach extensions from L to E of any element in $\mathfrak{L}$ coincide at x. $H(E,\mathfrak{L})$ is the largest subspace of E containing L to which every element in $\mathfrak{L}$ has a unique Hahn-Banach extension.

THEOREM 4. *Let $\mathfrak{L} \subseteq L^*$ be such that the set of $y^* \in F^*$ for which $(y^* \circ T)|L \in \mathfrak{L}$ and $\|y^*\| = \|y^*|M\|$ separates points of F. If T_1 is a linear operator from E into F such that $\|T_1\| = 1$ and $T_1|L = T|L$, then $T_1(x) = T(x)$ for any $x \in H(E,\mathfrak{L})$.*

PROOF. Let $x \in H(E,\mathfrak{L})$ and suppose, on the contrary, that $T_1(x) \neq T(x)$. By the hypothesis on F there exists a $y^* \in F^*$ such that $(y^* \circ T)|L \in \mathfrak{L}$. $\|y^*\| = \|y^*|M\|$ and $\langle T_1(x), y^* \rangle \neq \langle T(x), y^* \rangle$. Since $y^* \circ T$ and $y^* \circ T_1$ are Hahn-Banach extensions to E of the functional $(y^* \circ T)|L \in \mathfrak{L}$, they coincide at the point x, which contradicts the choice of y^*.

COROLLARY 5. *If $E = H(E,L^*)$ and the set of $y^* \in F^*$ with $\|y^*\| = \|y^*|M\|$ separates points of F, then T is uniquely determined by its values on L.*

The condition on F given in this corollary is satisfied if $F \subseteq M^{**}$, because M^* separates points of M^{**} and the norm of each $v^* \in M^*$ is the same as the norm of v^* as a linear functional on M^{**} (and so as a linear functional on F).

COROLLARY 6. *Suppose that* (i) *L has property U in E,* (ii) $E^*(L) = E^*$, (iii) *the set of $y^* \in F^*$ with $\|y^*\| = \|y^*|M\|$ separates points of F. Then T is an isometry and is determined by its values on L.*

EXAMPLE 1. Let Σ be a compact Hausdorff space and Σ_0 a closed subset of Σ. If $C(\Sigma)$ denotes the Banach space of continuous real (or complex) functions on Σ with the supremum norm, then the subspace $C(\Sigma|\Sigma_0) = \{f \in C(\Sigma): f|\Sigma_0 = 0\}$ has property U in $C(\Sigma)$ (cf. Phelps [7]). If $\Sigma \backslash \Sigma_0$ is dense in Σ, then it is easy to see that $C(\Sigma)$ is contained in the second dual of $C(\Sigma|\Sigma_0)$. If βN denotes the Čech compactification of the set N of positive integers, then (m) is isometrically isomorphic with $C(\beta N)$ and (c_0) with $C(\beta N|\beta N \backslash N)$. Hence (c_0) has property U in (m) and moreover $(m) = (c_0)^{**}$. So Rosenthal's result cited above follows from Corollaries 2 and 6. Concerning property U, we know that every two-sided ideal J of a C^*-algebra A has this property in A, of which the pair $\{(c_0), (m)\}$ is a special case. A much stronger result is contained in Dixmier [2, Proposition 2.11.7]. It is also seen that every hereditary subalgebra of a C^*-algebra A has property U in A.

EXAMPLE 2. We need neither the property U for L nor the fact like $E \subseteq L^{**}$ in order to assert that T is an isometry. This is illustrated by the well-known example in Korovkin's theory of approximation. Let $E = C([0,1])$ be the space of all continuous real (or complex) functions on the closed interval $[0, 1]$ and L the subspace of E spanned by three functions; 1, t and t^2. Although L does not have property U in E, every evaluation functional $\varepsilon_a: f \to f(a)$ with $a \in [0,1]$ on the space L has a unique Hahn-Banach extension, namely ε_a, to E. Since the set $\mathfrak{S} = \{\varepsilon_a: a \in [0,1]\}$ coincides with the set of extreme points of the unit ball of E^* and is contained in $S^*(E, L)$, we see that $E^*(L) = E^*$. So Corollary 2 applies to this case. On the other hand, we have $E = H(E, \mathfrak{S}|L)$. If the set of $y^* \in F^*$ for which $y^* \circ T = \varepsilon_a$ on L for some $a \in [0,1]$ separates points of F, then T is uniquely determined by its values on L. In particular, if T is a linear operator from $C([0,1])$ into itself with $\|T\| = 1$ and if $T(1) = 1$, $T(t) = t$ and $T(t^2) = t^2$, then T is the identity operator on $C([0,1])$.

EXAMPLE 3. Let $E = (m)$. Then the subspace (c_0) has a proper closed subspace L for which $E^*(L) = E^*$. Let $\{a_n: n = 1, 2, \ldots\}$ be a sequence such that, for each k, $|a_k| < \sum_{n \neq k} |a_n| < \infty$ and L the set of all sequences $x = (x_1, x_2, \ldots)$ in (c_0) with $\sum_{n=1}^{\infty} a_n x_n = 0$. It is clear that L is a closed subspace of (c_0). It is also easy to see that each evaluation functional $\varepsilon_n: x \to x_n$ has a unique Hahn-Banach extension to (c_0) and consequently to E. It follows that $E^*(L) = E^*$ and Corollary 2 applies to this L. L does not contain any nontrivial ideal of (c_0) (or (m)) and thus does not have property

U in (c_0) (or (m)). This last fact has been observed by S. Takahasi in a more general situation. It is seen also that L^{**} is strictly smaller than (m).

Here we include the following very slight modification of Kurtz's theorem [6, Theorem 3].

PROPOSITION 7. *Let $\mathfrak{F}$ be a subset of the unit sphere $\{y^* \in F^*: \|y^*\| = 1\}$ in F^* such that $y^* \circ T \in S^*(E, L)$ for any $y^* \in \mathfrak{F}$. Let $\{T_\lambda\}$ be a net of linear operators from E into F with $\|T_\lambda\| \leqslant 1$ such that $\|T_\lambda x - Tx\| \to 0$ for all $x \in L$. Then, for each $x \in E$, $\langle T_\lambda x, y^* \rangle \to \langle Tx, y^* \rangle$ uniformly on all $\sigma(F^*, F)$-compact subsets of $\mathfrak{F}$. In particular, if $\mathfrak{F}$ contains the $\sigma(F^*, F)$-closure of the extreme points of the unit sphere in F^*, then $\|T_\lambda x - Tx\| \to 0$ for each $x \in E$.*

2. **Simultaneous extensions.** Let Σ be a completely regular Hausdorff space and $C(\Sigma)$ the Banach space of all bounded continuous real (or complex) functions on Σ with the supremum norm. Let Ω be a subspace of Σ. If X and Y are subspaces of $C(\Omega)$ and $C(\Sigma)$, respectively, then a simultaneous extension is, by definition, a linear bounded operator T from X into Y such that $T(f)|\Omega = f$ for all $f \in X$. The result of Phelps cited above implies the following, which extends Banilower [1, Corollary 1.3].

PROPOSITION 8. *Let Σ be a completely regular Hausdorff space and Ω a locally compact subspace of Σ. Let $C_0(\Omega)$ be the subspace of $C(\Omega)$ consisting of elements f which vanish at infinity. If T is a linear operator from $C(\Omega)$ into $C(\Sigma)$ with $\|T\| = 1$ and $T|C_0(\Omega)$ is a simultaneous extension, then T is a simultaneous extension.*

PROOF. Let R: $C(\Sigma) \to C(\Omega)$ be the restriction operator. Then $R \circ T$ maps $C(\Omega)$ into $C(\Omega)$. Our assumption says that $\|R \circ T\| = 1$ and $R \circ T$ induces the identity operator on $C_0(\Omega)$. By Corollary 5, $R \circ T$ is the identity operator, as was to be proved.

The examples in §1 furnish other kinds of simultaneous extensions. For instance we have

PROPOSITION 9. *Let φ be a homeomorphism of $[0, 1]$ into a completely regular Hausdorff space Σ and T a linear operator from $C([0, 1])$ into $C(\Sigma)$ with $\|T\| \leqslant 1$. If T induces a simultaneous extension on the space spanned by 1, t, t^2 in the sense that $(Tf)(\varphi(t)) = f(t)$ for $f = 1, t, t^2$, then the same equality holds for any $f \in C([0, 1])$.*

Finally we extend [1, Proposition 1.4 and Theorem 1.5].

LEMMA 10. *Let Σ be a compact Hausdorff space and Ω a completely regular Hausdorff space which is extremally disconnected in the sense that the closure of every open set in Ω is open. If V is a linear isometric operator from $C(\Omega)$ into $C(\Sigma)$, then there exists a homeomorphism π from Ω into Σ such that, for any $f \in C(\Omega)$ and any $x \in \Omega$,*

$$|(V(1))(\pi(x))| = 1, \tag{1}$$

$$(2) \qquad (V(f))(\pi(x)) = f(x) \cdot (V(1))(\pi(x)).$$

PROOF. Since V is an isometry, the transposed mapping V^* of V maps the closed unit ball $B^*(\Sigma)$ of the dual $C(\Sigma)^*$ onto the closed unit ball $B^*(\Omega)$ of $C(\Omega)^*$. For each $x \in \Omega$ let ε_x be the evaluation functional $f \to f(x)$ on $C(\Omega)$. Since $C(\Omega)$ and $C(\beta\Omega)$ are isometrically isomorphic under the canonical mapping, we see that ε_x is an extreme point of the ball $B^*(\Omega)$. So the set $(V^*)^{-1}(\varepsilon_x) \cap B^*(\Sigma)$ ($= K(x)$, say) is a support of $B^*(\Sigma)$, which is convex and weakly* compact. Let μ be an extreme point of $K(x)$, which exists by the Krein-Milman theorem. Since $K(x)$ is a support of $B^*(\Sigma)$, μ is an extreme point of $B^*(\Sigma)$, so that there exist a point $y \in \Sigma$ and a number α, $|\alpha| = 1$, satisfying $\mu = \alpha^{-1}\varepsilon_y$ in view of [4, Lemma 7]. Thus, for each $x \in \Omega$, there exist a point $y \in \Sigma$ and a number α with $|\alpha| = 1$ such that

$$\begin{aligned}(V(f))(y) &= \langle V(f), \varepsilon_y \rangle = \langle f, V^*(\varepsilon_y) \rangle = \langle f, \alpha V^*(\mu) \rangle \\ &= \langle f, \alpha\varepsilon_x \rangle = \alpha f(x)\end{aligned}$$

for all $f \in C(\Omega)$. We define, for each $x \in \Omega$, $\psi(x)$ to be the set of all $y \in \Sigma$ such that there exists a number $\alpha(y)$ with $|\alpha(y)| = 1$ satisfying $(V(f))(y) = \alpha(y)f(y)$ for all $f \in C(\Omega)$. It is easy to see that $\psi(x)$ is closed for every $x \in \Omega$ and the mapping ψ: $x \to \psi(x)$ is an upper semicontinuous mapping from Ω into the family of nonvoid compact subsets of Σ, i.e., $\{x \in \Omega: \psi(x) \subseteq \Sigma'\}$ is open in Ω if Σ' is open in Σ. By use of a continuous selection theorem [5, Theorem 1.1] we can find a continuous mapping π from Ω into Σ such that $\pi(x) \in \psi(x)$ for any $x \in \Omega$. Since the subsets $\psi(x)$ are mutually disjoint, π is one-to-one. We have shown that, for any $f \in C(\Omega)$ and any $x \in \Omega$, $(V(f))(\pi(x)) = \alpha(\pi(x))f(x)$. If $f \equiv 1$, then we have $(V(1))(\pi(x)) = \alpha(\pi(x))$ for any $x \in \Omega$. Since $|\alpha(y)| = 1$, we have proved the statements (1) and (2). Finally let $\{x_\lambda\}$ be a net in Ω, $x \in \Omega$ and suppose that $\pi(x_\lambda)$ tend to $\pi(x)$. Then (2) implies that $f(x_\lambda)$ tend to $f(x)$ for any $f \in C(\Omega)$. Since Ω is completely regular, we see that $x_\lambda \to x$ in Ω. Hence π is a homeomorphism.

THEOREM 11. *Let Σ be a compact Hausdorff space and Ω an extremally disconnected, completely regular Hausdorff space. Then $C(\Sigma)$ contains a subspace isometrically isomorphic to $C(\Omega)$ if and only if there exists a subspace Σ_0 of Σ such that Σ_0 is homeomorphic with Ω and there is a simultaneous extension T from $C(\Sigma_0)$ into $C(\Sigma)$ with norm one.*

PROOF. We have only to prove the necessity of the theorem. Let V be a linear isometric mapping from $C(\Omega)$ into $C(\Sigma)$. Then there exists a homeomorphism π from Ω into Σ satisfying the condition of the preceding lemma. We set $\Sigma_0 = \pi(\Omega)$. Define Q: $C(\Sigma_0) \to C(\Omega)$ by setting

$$(Q(g))(x) = g(\pi(x))/(V(1))(\pi(x)).$$

We see that Q is a linear isometry from $C(\Sigma_0)$ onto $C(\Omega)$ and therefore that

$T = V \circ Q$ is an isometry from $C(\Sigma_0)$ into $C(\Sigma)$, which is easily seen to be a simultaneous extension from $C(\Sigma_0)$ into $C(\Sigma)$, as was to be proved.

This theorem is reduced to [1, Theorem 1.5], when Ω is the discrete space of positive integers.

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