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SECOND ORDER SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. Some existence theorems are established for nonlinear second order systems with nonlinear two point boundary conditions. The system is assumed to satisfy certain differential inequality conditions at the boundary of a region.

Let R^n be n-dimensional Euclidean space with scalar product $x \cdot y$ for x and y in R^n and norm $\|x\| = \sqrt{x \cdot x}$. In the following discussion, f: $[a,b] \times R^n \times R^n \to R^n$, where $[a,b]$ is an interval of real numbers, will be assumed continuous. The system

$$x'' = f(t, x, x'), \tag{1}$$

will be considered together with boundary conditions of the form

$$x(a) - A_1 x'(a) = 0, \tag{2}$$

$$x(b) + A_2 x'(b) = 0, \tag{3}$$

or

$$B_1 x(a) - x'(a) = 0, \tag{4}$$

$$B_2 x(b) + x'(b) = 0, \tag{5}$$

where A_1, A_2, B_1, B_2 are continuous functions from R^n to R^n.

Lasota and Yorke [3] have proved existence theorems for (1), (2), (3) in the case where A_1 and A_2 are linear and positive definite, using Lyapunov-type inequalities for f. We show in this paper that similar results are true under suitable conditions on the boundary functions even if these functions are nonlinear and differential inequality conditions on f are assumed to hold only at the boundary of a region. Nonlinear boundary conditions have been studied in the scalar case by a variety of authors, including Bebernes and Wilhelmsen [1], and Jackson and Klaasen [2].

Suppose r: $R^n \to R$ is of class C^2 with gradient u: $R^n \to R^n$ and Hessian

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$P\colon R^n \to \mathfrak{L}(R^n, R^n)$. Let the first and second derivatives of r with respect to (1) be denoted by

$$r' = u \cdot x', \quad r''_f = x'P \cdot x' + u \cdot f.$$

Auxiliary functions of this type have been used by Mawhin [4] to study (1), (2), (3) in the case $A_1 = A_2 = 0$.

We say that f satisfies condition (N) with respect to r if there is a positive, nondecreasing, continuous function ϕ defined on $(0, \infty)$ such that $s^2/\phi(s) \to \infty$ as $s \to \infty$ and $\|f(t,x,y)\| \leqslant \phi(\|y\|)$ for $t \in [a,b]$, $r(x) \leqslant 0$, $y \in R^n$. This is a Nagumo condition on f of the type used by Schmitt and Thompson [5].

THEOREM 1. *Let r be as above and assume $\{x\colon r(x) < 0\}$ is bounded and contains* 0. *Suppose $u(x) \neq 0$ whenever $r(x) = 0$ and*

$$\inf\{(u(x) \cdot x)/\|u(x)\|\,\|x\|\colon r(x) = 0\} = \alpha \geqslant 0. \tag{6}$$

Assume f satisfies condition (N) *with respect to r and*

$$r''_f \geqslant 0 \quad \textit{when } r = 0 \textit{ and } r' = 0. \tag{7}$$

Suppose $\{y\colon r(A_1 y) < 0\}$ is bounded and contains 0, *and*

$$\inf\{(y \cdot A_1 y)/\|y\|\,\|A_1 y\|\colon r(A_1 y) = 0\} \geqslant \sqrt{1 - \alpha^2}, \tag{8}$$

(9)
$$\textit{if } D = \{y\colon r(-A_2 y) = 0, \textit{ or } r(A_1 y) = 0 \textit{ and } A_2 y \neq 0\} \neq \varnothing,$$
$$\textit{then } \inf\{(y \cdot A_2 y)/\|y\|\,\|A_2 y\|\colon y \in D\} > \sqrt{1 - \alpha^2}.$$

Then there is at least one solution x of (1), (2), (3) *with $r(x(t)) \leqslant 0$ for $t \in [a,b]$.*

PROOF. We assume initially that f satisfies

$$r''_f > 0 \quad \text{when } r = 0 \text{ and } r' = 0, \tag{7$'$}$$

and that initial value problems for (1) have unique solutions.

Define $W = \{(x,y) \in R^n \times R^n\colon x = A_1 y\}$ and $V = \{(x,y) \in W\colon r(x) < 0\}$. Note that $V \neq \varnothing$ since $(A_1 0, 0) \in V$. We now define a function $T\colon \overline{V} \to R^n$. If $(x,y) \in \overline{V}$, then there is a unique solution $x(t)$ of (1) with $(x(a), x'(a)) = (x,y)$. Since by Lemma 2.1 of [5] the derivative $x'(t)$ is bounded as long as $r(x(t)) \leqslant 0$, the extension theorem for ordinary differential equations implies that the trajectory $(t, x(t), x'(t))$ extends to the boundary of $[a,b] \times \{x\colon r(x) \leqslant 0\} \times R^n$. Let t_0 be the smallest value of t for which $r(x(t)) = 0$, or, if there is no such t, let $t_0 = b$. Define $T(x,y) = x(t_0) + A_2 x'(t_0)$.

We claim that T is continuous on $\overline{V}$. Let $(x,y) \in \overline{V}$ and $(x_k, y_k) \in \overline{V}$ for $k = 1, 2, \ldots$, and suppose $(x_k, y_k) \to (x,y)$ as $k \to \infty$. Let x and x_k be the solutions of (1) with $(x(a), x'(a)) = (x, y)$ and $(x_k(a), x_k'(a)) = (x_k, y_k)$ for $k = 1, 2, \ldots$.

Suppose $r(x(t)) < 0$ for $t \in [a, b]$. By the standard convergence theorem, $r(x_k(t)) < 0$ for k sufficiently large and $t \in [a, b]$. Since $(x_k(b), x_k'(b)) \to (x(b), x'(b))$ as $k \to \infty$, we conclude $T(x_k, y_k) \to T(x, y)$ as $k \to \infty$ in this case.

Otherwise, there is a smallest t_0 so that $r(x(t_0)) = 0$. Suppose $a < t_0 < b$. Let $s(t) = r(x(t))$ for all t in the domain of x. Since $s(t) < 0$ for $a \leqslant t < t_0$ and $s(t_0) = 0$, we have $s'(t_0) \geqslant 0$. If $s'(t_0) > 0$, there is a $\delta > 0$ so that $s(t) > 0$ for $t_0 < t \leqslant t_0 + \delta$. If $s'(t_0) = 0$, then by (7)' $s''(t_0) > 0$. This is impossible because $s(t) < 0$ for $a \leqslant t < t_0$.

Fix $\lambda \in (0, \delta)$. Let

$$\varepsilon = \inf\{|s(t)|: t \in [0, t_0 - \lambda] \cup [t_0 + \lambda, t_0 + \delta]\}.$$

Choose k sufficiently large so that $|r(x(t)) - r(x_k(t))| < \varepsilon/2$, $\|x_k(t) - x(t)\| < \lambda$ and $\|x_k'(t) - x'(t)\| < \lambda$ for $t \in [0, t_0 + \delta]$. For these values of k, $r(x_k(t)) < -\varepsilon/2 < 0$ for $t \in [0, t_0 - \lambda]$ and $r(x_k(t)) > \varepsilon/2 > 0$ for $t \in [t_0 + \lambda, t_0 + \delta]$. Hence for each such k there is a smallest t, say t_k, with $r(x_k(t_k)) = 0$ and $t_k \in (t_0 - \lambda, t_0 + \lambda)$. Furthermore, $\|x_k(t_k) - x(t_0)\|$ and $\|x_k'(t_k) - x'(t_0)\|$ are no larger than the diameter of the set

$$\{(t, x, y): t_0 - \lambda < t < t_0 + \lambda, \|x - x(t)\| < \lambda, \|y - x'(t)\| < \lambda\}.$$

Since these diameters approach 0 as $\lambda \to 0$, we conclude $T(x_k, y_k) \to T(x, y)$ as $k \to \infty$ in this case.

In case $t_0 = b$, the above argument can be modified slightly to obtain $T(x_k, y_k) \to T(x, y)$ as $k \to \infty$. Finally, consider the possibility $t_0 = a$. We have $T(x, y) = x(a) + A_2 x'(a)$ and $r(x(a)) = 0$. By (6), the angle θ_1, between $u(x(a))$ and $x(a)$ has $\cos\theta_1 \geqslant \alpha$, and by (8), the angle θ_2 between $x'(a)$ and $x(a) = A_1 x'(a)$ has $\cos\theta_2 \geqslant \sqrt{1 - \alpha^2}$. Thus $\cos^2\theta_1 + \cos^2\theta_2 \geqslant 1$ and $\theta_1 + \theta_2 \leqslant \pi/2$. By a well-known result from geometry, the angle between $u(x(a))$ and $x'(a)$ is less than or equal to $\theta_1 + \theta_2$, so $u(x(a)) \cdot x'(a) \geqslant 0$. It follows that there is a $\delta > 0$ so that $r(x(t)) > 0$ for $a < t \leqslant t_0 + \delta$, and the remainder of the earlier argument can again be modified to conclude $T(x_k, y_k) \to T(x, y)$ as $k \to \infty$. Thus T is continuous on $\overline{V}$.

Define $H: R^n \to W$ by $H(y) = (A_1 y, y)$. Note that H is a homeomorphism. Let $U = H^{-1}(V)$. Then U is bounded, open, nonempty and its boundary $\partial U \subseteq \{y: r(A_1 y) = 0\}$. We will calculate the Brouwer degree for the map $T \circ H: \overline{U} \to R^n$. See Schwartz [6] for a discussion of the Brouwer degree and its properties.

Consider the maps $\lambda T \circ H + (1 - \lambda)I$ for $0 \leqslant \lambda \leqslant 1$ on $\overline{U}$, where I is the identity. Let $y \in \partial U$ and suppose $\lambda T \circ Hy + (1 - \lambda)y = 0$. If $\lambda \neq 0$, we have $T \circ Hy = -((1 - \lambda)/\lambda)y$, i.e., $(A_1 + A_2)y = -((1 - \lambda)/\lambda)y$. Note also that $r(A_1 y) = 0$ and $y \neq 0$. If $A_2 y = 0$, then $A_1 y = -((1 - \lambda)/\lambda)y$ contradicts (8). If $A_2 y \neq 0$, then (8) and (9) together imply $y \cdot (A_1 + A_2)y > 0$, which contradicts $(A_1 + A_2)y = -((1-\lambda)/\lambda)y$. If $\lambda = 0$, then $y = 0$, another contradiction. Thus the maps $\lambda T \circ H + (1-\lambda)I$ do not vanish on ∂U for $0 \leqslant \lambda \leqslant 1$. The Brouwer degree $d[T \circ H, U, 0]$ exists, and by invariance of degree under homotopy, $d[T \circ H, U, 0] = d[I, U, 0] = 1$ since $0 \in U$. Thus there is a $y \in U$ so that $T \circ Hy = 0$. This means that there is a solution $x(t)$ of (1) with $(x(a), x'(a)) \in V$ and $T(x(a), x'(a)) = x(t_0) + A_2 x'(t_0) = 0$.

Suppose that $r(x(t_0)) = 0$. Since $t_0 > a$, $u(x(t_0)) \cdot x'(t_0) \geqslant 0$. Let ϕ_1 be the angle between $u(x(t_0))$ and $x(t_0)$, ϕ_2 the angle between $x'(t_0)$ and $A_2 x'(t_0)$ and ϕ_3 the angle between $u(x(t_0))$ and $x'(t_0)$. Then $\phi_3 \leqslant \pi/2$, and by (6), (9), $\cos^2\phi_1 + \cos^2\phi_2 > 1$, so $\phi_1 + \phi_2 < \pi/2$. Now the angle between $A_2 x'(t_0)$ and $x(t_0)$ is no larger than $\phi_1 + \phi_2 + \phi_3$, the sum of which is less than π, contradicting $A_2 x'(t_0) = -x(t_0)$. Thus $r(x(t_0)) \neq 0$, and it follows that $t_0 = b$. We conclude that x is a solution of (1), (2), (3) with $r(x(t)) < 0$ for $a \leqslant t \leqslant b$.

Now suppose initial value problems of (1) do not necessarily have unique solutions. We show first that $yP(x) \cdot y \geqslant 0$ whenever $r(x) = 0$ and $u(x) \cdot y = 0$. Suppose, on the contrary, that there is an x with $r(x) = 0$ and a $y \in R^n$ with $\|y\| = 1$, $u(x) \cdot y = 0$ and $yP(x) \cdot y < 0$. For all constants c, $r'(cy) = u \cdot cy = 0$, so by (7)',

$$r''_f = c^2 yP(x) \cdot y + u(x) \cdot f(t, x, cy) > 0. \tag{10}$$

Choose c large enough that $c^2/\phi(c) \geqslant \|u(x)\|/|yP(x) \cdot y|$. Then by condition (N),

$$|u(x) \cdot f(t, x, cy)| \leqslant \|u(x)\| \, \|f(t, x, cy)\| \leqslant \|u(x)\| \phi(c) \leqslant c^2 |yP(x) \cdot y|,$$

contradicting (10).

By Lemma 2.1 in [5], there is a $\mu > 0$ so that if $x(t)$ satisfies $\|x''\| \leqslant \phi(\|x'\|)$ and $r(x(t)) \leqslant 0$ for $t \in [a, b]$, then $\|x'(t)\| \leqslant \mu$ $(a \leqslant t \leqslant b)$. Define for all $(t, x) \in [a, b] \times R^n$,

$$F(t, x, y) = \begin{cases} f(t, x, y), & \|y\| \leqslant \mu, \\ f(t, x, \mu y/\|y\|), & \|y\| > \mu. \end{cases}$$

Fix (x, y) so that $r(x) = 0$ and $r'(x, y) = 0$. If $\|y\| \leqslant \mu$, then $r''_F - r''_f > 0$ at (x, y). If $\|y\| > \mu$, then

$$\begin{aligned} r''_F &= yP(x) \cdot y + u(x) \cdot f(t, x, \mu y/\|y\|) \\ &= (\|y\|^2/\mu^2)((\mu y/\|y\|)P(x) \cdot \mu y/\|y\|) + u \cdot f(t, x, \mu y/\|y\|) \\ &\geqslant (\mu y/\|y\|)P(x) \cdot \mu y/\|y\| + u \cdot f(t, x, \mu y/\|y\|) > 0, \end{aligned}$$

since $r''_f > 0$. Thus F satisfies (7)'.

Choose a sequence of C^1 functions $\{F_n\}_{n=1}^{\infty}$ which converges uniformly to F on $[a, b] \times \{x: r(x) \leqslant 0\} \times R^n$. For n sufficiently large, F_n satisfies (7)' and condition (N), so we have a sequence $\{x_n\}$ so that each x_n satisfies $x'' = F_n(t, x, x')$, (2), (3) and $r(x(t)) < 0$ for $t \in [a, b]$. Some subsequence of $\{x_n\}$ converges to a solution x of $x'' = F(t, x, x')$, (2), (3) with $r(x(t)) \leqslant 0$ for $a \leqslant t \leqslant b$. Since F satisfies $\|F(t, x, y)\| \leqslant \phi(\|y\|)$ for $r(x(t)) \leqslant 0$ and $y \in R^d$, we have $\|x'(t)\| \leqslant \mu$ for $t \in [a, b]$, so x is a solution of (1).

Finally, if f satisfies (7), then $f_\varepsilon = f + \varepsilon u$ ($\varepsilon > 0$) satisfies (7)′, so the above results apply to f_ε, and the proof is completed by letting $\varepsilon \to 0$. Q.E.D.

Of particular interest is the case that both A_1 and A_2 are linear. The assumption that $\{y: r(A_1 y) < 0\}$ is bounded requires A_1 to be nonsingular, and (8) requires, in addition, that for all $y \neq 0$, the angle between y and $A_1 y$ has cosine at least $\sqrt{1 - \alpha^2}$. Note that if $y \in \operatorname{kernel} A_2$, then $y \notin D$. Thus (9) dictates that the cosine of the angle between y and $A_2 y$ is greater than $\sqrt{1 - \alpha^2}$ for all $y \notin \ker A_2$. The most elementary choice for r is $r(x) = \|x\|^2 - R^2$, where $R > 0$. Then (7) becomes

$$(11) \qquad \|y\|^2 + x \cdot f(t, x, y) \geqslant 0 \quad \text{whenever } \|x\| = R \text{ and } x \cdot y = 0,$$

and (6) is satisfied with $\alpha = 1$, so one requires that A_1 be nonsingular and nonnegative definite and $A_2 y \cdot y > 0$ for all $y \notin \ker A_2$.

COROLLARY. *Assume A_1 and A_2 are linear and f satisfies condition* (N) *with respect to $r(x) = \|x\|^2 - R^2$ for some $R > 0$. Suppose that* (11) *is satisfied, A_1 is nonsingular, $A_1 y \cdot y \geqslant 0$ for all $y \in R^n$ and $A_2 y \cdot y > 0$ for all $y \notin \ker A_2$. Then there is at least one solution x of* (1), (2), (3) *with $\|x(t)\| \leqslant R$ for $a \leqslant t \leqslant b$.*

The next theorem provides criteria for the existence of a solution to (1), (4), (5). A proof can be given which is almost identical to the proof of Theorem 1.

THEOREM 2. *Let r be as above and assume $\{x: r(x) < 0\}$ is bounded and contains* 0. *Suppose $u(x) \neq 0$ whenever $r(x) = 0$, and* (6) *is satisfied. Assume f satisfies condition* (N) *with respect to r and* (7) *holds. Suppose*

$$(12) \qquad \inf\{(x \cdot B_1 x)/\|x\|\,\|B_1 x\| : r(x) = 0 \text{ and } B_1 x \neq 0\} \geqslant \sqrt{1 - \alpha^2},$$

$$(13) \qquad \begin{aligned} &B_2 x \neq 0 \text{ whenever } r(x) = 0 \text{ and} \\ &\inf\{(x \cdot B_2 x)/\|x\|\,\|B_2 x\| : r(x) = 0\} > \sqrt{1 - \alpha^2}. \end{aligned}$$

Then there is at least one solution x of (1), (4), (5) *with $r(x(t)) \leqslant 0$ for $t \in [a, b]$.*

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