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RATIONAL SINGULARITIES OF HIGHER DIMENSIONAL SCHEMES

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ABSTRACT. Two examples of rational singularities of schemes over an algebraically closed field of characteristic zero are given: Singularities occurring as the quotient of a regular scheme by a finite group and singularities of the type $u^2 - v^2 - g(t_1, \ldots, t_N)$.

Unless otherwise explicitly mentioned, we assume that all schemes are irreducible, reduced and of finite type over an algebraically closed field k of characteristic zero and that all points are k-rational.

Let U be a scheme, X a regular scheme and $g: X \to U$ a proper, surjective, birational morphism. We call $g: X \to U$ a *resolution* of U.

DEFINITION 1. Let Y be a scheme and $y \in Y$. Then Y has a rational singularity at y if there exists a neighbourhood U of y in Y, such that for every resolution $g: X \to U$ we have $g_*\mathcal{O}_X \cong \mathcal{O}_U$ and $R^ig_*\mathcal{O}_X = 0$ for $i \neq 0$ (henceforth we write $\mathbf{R}g_*\mathcal{O}_X \cong \mathcal{O}_U$).

In this paper we want to discuss two examples of rational singularities needed in [5, §5].

REMARKS. (i) Using "flat base change" [1] it is easy to see that the question whether $y \in Y$ is a rational singularity depends only on $\hat{\mathcal{O}}_{y,Y}$ ("$\hat{}$" denotes the completion with respect to the maximal ideal).

(ii) If the base field has positive characteristic, one needs additional conditions to define rational singularities [3].

(iii) Every regular point of a scheme is a rational singularity [2].

(iv) In Definition 1 it is sufficient to consider one resolution of U.

This last statement follows from (iii) and the Leray spectral sequence. Using the same kind of argument one gets

LEMMA 1. *Let $h: Y' \to Y$ be a proper, surjective, birational morphism of schemes. Assume that Y' has only rational singularities; then $\mathbf{R}h_*\mathcal{O}_{Y'} \cong \mathcal{O}_Y$, if and only if Y has only rational singularities.*

We first consider quotient singularities:

DEFINITION 2. Let Y be a scheme and $y \in Y$. Then Y has a *quotient singularity* at y if there exist a regular scheme Y' and a finite group G acting

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on Y' such that Y'/G exists and is isomorphic to some neighbourhood of y in Y.

Let W be a regular scheme and $D \subseteq W$ a closed, reduced subscheme of codimension 1. We say that D has *normal crossings* if the irreducible components of D are regular and if for every point $w \in W$ regular parameters $x_1, \ldots, x_n$ exist such that D is defined by $x_1 \cdot x_2 \cdot \ldots \cdot x_r = 0$.

LEMMA 2. *Let W be a regular scheme and f: $Y \to W$ a finite morphism of normal schemes. Assume that the ramification locus $\Delta(Y/W)$ (see [4]) has only normal crossings. Then Y has quotient singularities and f is flat.*

PROOF. The first statement follows from Abhyankar's lemma: The proof is exactly that used in [4, pp. 32–33] to prove "Satz 4.1". One must simply add "and hence P_2 is a regular point of X_2" after the 18th line of p. 33.

Since flatness is a local property, we may assume that for some regular scheme Y' and for some group G we have h: $Y' \to Y'/G \cong Y$. Then $f \cdot h$ is a finite morphism of regular schemes, and hence $(f \cdot h)_* \mathcal{O}_{Y'}$ is locally free. Let η: $\mathcal{O}_Y \to h_* \mathcal{O}_{Y'}$ be the map "multiplication with $\operatorname{ord}(G)^{-1}$", and Tr: $h_* \mathcal{O}_{Y'} \to \mathcal{O}_Y$ the trace map. Then $\mathrm{Tr} \cdot \eta$ is an isomorphism and, therefore, $f_* \mathcal{O}_Y$ a direct summand of $(f \cdot h)_* \mathcal{O}_{Y'}$.

PROPOSITION 1. *Every quotient singularity is a rational singularity.*

PROOF. Assume that for some $n \geqslant 1$ and every scheme Y having quotient singularities we already know:

For every resolution g: $X \to Y$ we have $R^i g_* \mathcal{O}_X = 0$ for $0 < i < n$.

Let $Y = Y'/G$ (as in Definition 2) and g: $X \to Y$ be a resolution. Denote the normalization of X in the function field of Y' by X' and choose a resolution h': $W' \to X'$ of X'. We denote the natural morphisms by:

$$\begin{tikzcd} W' \arrow[r, "h'"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & Y' \arrow[d, "f"] \\ & X \arrow[r, "g"] & Y \end{tikzcd}$$

Using "embedded resolution of singularities" [2] we may assume that $\Delta(X'/X)$ has normal crossings. X' also has quotient singularities (Lemma 2) and, by assumption, $R^i h'_* \mathcal{O}_{W'} = 0$ for $0 < i < n$. The Leray spectral sequence gives an injection $R^n g'_* (h'_* \mathcal{O}_{W'}) \to R^n (g' \cdot h')_* \mathcal{O}_{W'}$. Since $g' \cdot h'$ is a resolution of a regular scheme, $R^n (g' \cdot h')_* \mathcal{O}_{W'} = 0$ and therefore $0 = f_* R^n g'_* \mathcal{O}_{X'} = R^n g_* (f'_* \mathcal{O}_{X'})$. Since $\mathcal{O}_X$ is a direct summand of $f'_* \mathcal{O}_{X'}$ (proof of Lemma 2), we get $R^n g_* \mathcal{O}_X = 0$.

The second example is given by an explicit equation:

PROPOSITION 2. *Let Y be a scheme and $y \in Y$ such that*

$$\hat{\mathcal{O}}_{y,Y} = k[[t_1, \ldots, t_n, u, v]] / \left(u^2 - v^2 - g(t_1, \ldots, t_n)\right),$$

$0 \neq g(t_1, \ldots, t_n) \in k[[t_1, \ldots, t_n]]$. *Then we have*:

(i) $\hat{\mathcal{O}}_{y,Y}$ *(and hence* $\mathcal{O}_{y,Y}$*) is Cohen-Macaulay and normal.*

(ii) $\hat{\mathcal{O}}_{y,Y}$ *is flat over* $k[[t_1, \ldots, t_n]]$.

(iii) y *is a rational singularity of* Y.

PROOF. $\hat{\mathcal{O}}_{y,Y}$ is a complete intersection and, hence, Cohen-Macaulay. The normality follows from Serre's criterion.

(ii) is true since the induced morphism of schemes is equidimensional. We may assume that

$$Y = \operatorname{Spec}\Big(k\big[\big[t_1, \ldots, t_n\big]\big]\big[u, v\big] / \big(u^2 - v^2 - g(t_1, \ldots, t_n)\big)\Big)$$

and $W = \operatorname{Spec}(k[[t_1, \ldots, t_n]])$. Let D be the subscheme of W defined by $g(t_1, \ldots, t_n) = 0$. Using "embedded resolution of singularities" [2], "flat base change" [1] and Lemma 1, we may assume that D_{red} has normal crossings. After choosing another system of regular parameters in W, we get $g(t_1, \ldots, t_n) = t_1^{\nu_1} \cdot t_2^{\nu_2} \cdot \ldots \cdot t_n^{\nu_n}$, $\nu_i \in \mathbf{N}$. Although singularities of this type are known to be rational [3], we prove it directly:

Step 1. Blow up ideals of the form $\langle u, v, t_i \rangle$ to decrease the ν_i's until $\nu_i = 1$ or $\nu_i = 0$ for $i = 1, \ldots, n$.

Step 2. Blow up ideals of the form $\langle u, v, t_i, t_j \rangle$, $i \neq j$, to decrease the number of variables occurring in $g(t_1, \ldots, t_n)$.

Denote one of the morphisms in Step 1 or Step 2 by $h\colon X \to Y$ and the exceptional locus by E. It is easy to see $H^q(E, \mathcal{O}_E(p)) = 0$ for $q > 0$ and $p \geqslant 0$. Using decreasing induction on p and the exact sequence

$$0 \to \mathcal{O}_X(p + 1) \to \mathcal{O}_X(p) \to \mathcal{O}_E(p) \to 0,$$

we obtain $\mathbf{R}h_* \mathcal{O}_X \cong \mathcal{O}_Y$.

REMARKS. (i) The type of singularities discussed in Proposition 2 occurs in stable curves over a regular base scheme. Using flat base change, it follows that a stable curve over a scheme with rational singularities has rational singularities itself. (ii) One example: Surprisingly $u^2 - v^2 - t_1^3 - t_2^7$ defines a rational singularity while $u^2 - t_1^3 - t_2^7$ does not.

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