REMARK ON MEASURABLE GRAPH THEOREMS

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Abstract. A theorem generalizing some known measurable graph theorems is proved.

1. Notations. Two measurable spaces \((X_1, \mathcal{E}_1)\) and \((X_2, \mathcal{E}_2)\) are isomorphic if there exists a bimeasurable one-to-one function from \(X_1\) onto \(X_2\). A measurable space is called a standard measurable space if it is isomorphic with \((I, \mathcal{B}(I))\), the unit interval and its Borel \(\sigma\)-field. We call it a Blackwell space if it is isomorphic with some space \((A, \mathcal{B}(A))\), where \(A\) is an analytic subset of \(I\) and \(\mathcal{B}(A)\) is the system of all intersections of \(A\) with sets in \(\mathcal{B}(I)\). We claim that \((A', \mathcal{B}(X))\) is a standard measurable space if \(X\) is any nondenumerable complete separable metric space and \(\mathcal{B}(X)\) its Borel \(\sigma\)-field. If \((X, \mathcal{E}, \mu)\) is a measure space, then we write throughout the paper \(\mathcal{E}_\mu\) for the completion of \(\mathcal{E}\) with respect to \(\mu\).

2. Main results. Let \((X, \mathcal{E})\) and \((Y, \mathcal{B})\) be measurable spaces, \(f\) a function \(f: X \rightarrow Y\), \(\text{gr}(f)\) the graph of \(f\) and \(\mathcal{E} \otimes \mathcal{B}\) the product \(\sigma\)-field. It is well known that \((\mathcal{E}, \mathcal{B})\)-measurability of \(f\) implies \(\text{gr}(f) \in \mathcal{E} \otimes \mathcal{B}\) if \(\mathcal{B}\) is countably separated (see e.g. [5, 1.3.4 and 5]). Concerning the reverse implication we shall prove in this paper two notes:

(A) Let \((X, \mathcal{E})\) and \((Y, \mathcal{B})\) be Blackwell spaces and \(\mu\) be a measure on \(\mathcal{E}\). If \(\text{gr}(f) \in \mathcal{E}_\mu \otimes \mathcal{B}\), then \(f\) is \((\mathcal{E}_\mu, \mathcal{B})\)-measurable.

(B) Let \((X, \mathcal{E}, \mu)\) be a \(\sigma\)-finite measure space and \((Y, \mathcal{B})\) a Blackwell space. If \(\text{gr}(f) \in \mathcal{E}_\mu \otimes \mathcal{B}\), then \(f\) is \((\mathcal{E}_\mu, \mathcal{B})\)-measurable.

In [3] Buckley has proved a result corresponding to (A) on the more restrictive assumption that the spaces \((X, \mathcal{E})\) and \((Y, \mathcal{B})\) are complete separable metric spaces with their Borel \(\sigma\)-fields. In the most important case of a \(\sigma\)-finite measure, Buckley’s result is known from Bierlein’s measurable graph theorem in [2], where \((Y, \mathcal{B})\) is assumed to be a standard measurable space and \((X, \mathcal{E}, \mu)\) is any \(\sigma\)-finite measure space. In (B) Bierlein’s assumption on \((Y, \mathcal{B})\) is generalized to the assumption of a Blackwell space.

Now (A) and (B) may be applied to give a more general version of Buckley’s theorem, of the main part of Bierlein’s measurable graph theorem and of the known fact that \(\text{gr}(f) \in \mathcal{E} \otimes \mathcal{B}\) is equivalent to \((\mathcal{E}, \mathcal{B})\)-measurability.

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bility of $f$ if both spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are Blackwell spaces (cf. [5, II.4.3]). Since $\mathcal{B}$ is countably separated if $(Y, \mathcal{B})$ is a Blackwell space, the following theorem is a consequence of (A) and (B).

**Measurable Graph Theorem.** Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a Blackwell space and $f$ a function $f: X \to Y$. If either the measure $\mu$ is $\sigma$-finite or the space $(X, \mathcal{A})$ is a Blackwell space, then the equivalence holds:

$f$ is $(\mathcal{A}_\mu, \mathcal{B})$-measurable if and only if $\text{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$.

3. **Proofs.** The proof of (A) is based on the following lemma. The proof of the lemma is omitted as a standard one.

**Lemma.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $(Y, \mathcal{B})$ a measurable space. If $G \in \mathcal{A}_\mu \otimes \mathcal{B}$, then there exists a set $N_0 \in \mathcal{A}$ such that $\mu(N_0) = 0$ and $G \setminus (N_0 \times Y) \in \mathcal{A} \otimes \mathcal{B}$.

**Proof of (A).** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be Blackwell spaces and $\text{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$. Then, by the Lemma, $X$ is the union of two disjoint sets $X_0 \in \mathcal{A}$ and $N_0 \in \mathcal{A}$ such that

$$\mu(N_0) = 0 \quad \text{and} \quad \text{gr}(f) \setminus (N_0 \times Y) \in \mathcal{A} \otimes \mathcal{B}.$$ 

Let $\mathcal{A} \cap X_0$ be the $\sigma$-field of all intersections of $X_0$ with sets of $\mathcal{A}$. Since, by definition, $(X_0, \mathcal{A} \cap X_0)$ is a Blackwell space, too, we can apply the measurable graph theorem for Blackwell spaces (see [5, II.4.3]) to the function $f_0: X_0 \to Y$ defined as the restriction of $f$ on $X_0$. Since $\text{gr}(f_0) = \text{gr}(f) \setminus (N_0 \times Y)$ is an element of $(\mathcal{A} \cap X_0) \otimes \mathcal{B}$, by this theorem, $f_0$ is an $(\mathcal{A} \cap X_0, \mathcal{B})$-measurable function. Then we have for any $B \in \mathcal{B}$,

$$f^{-1}(B) = (f^{-1}(B) \cap X_0) \cup (f^{-1}(B) \cap N_0)$$

$$= f_0^{-1}(B) \cup (f^{-1}(B) \cap N_0) \in \mathcal{A}_\mu,$$

because $f_0^{-1}(B) \in \mathcal{A}$, $f^{-1}(B) \cap N_0$ is a subset of $N_0$ and $\mu(N_0) = 0$. This shows that $f$ is $(\mathcal{A}_\mu, \mathcal{B})$-measurable, and (A) is proved.

To prove (B) it is possible to apply a projection theorem (II.2.2 in [5] is a suitable one) and Choquet's theorem on capacities (see [4, Theorem 1]) as done by Bierlein [2] in the case of a standard measurable space $(Y, \mathcal{B})$. However, to give a short proof, we claim that (B) is an immediate consequence of the following generalization of Aumann's selection theorem (see [1, p. 17]) proved in [6] (cf. Theorem 3, and for the difference in formulation see [5, III.2.3]):

**Selection Theorem.** Let $(X, \mathcal{A})$ be a measurable space having the property that $\mathcal{A}$ equals its universal completion, $(Y, \mathcal{B})$ be a Blackwell space and $C \in \mathcal{A} \otimes \mathcal{B}$. If the projection of $C$ on $X$ is all of $X$, then there exists an $(\mathcal{A}, \mathcal{B})$-measurable function $s: X \to Y$ such that $(x, s(x)) \in C$ for all $x \in X$.

theorem generalizes also the known result that for Blackwell spaces \((X, \mathcal{G})\) and \((Y, \mathcal{B})\), \(\text{gr}(f) \in \mathcal{G} \otimes \mathcal{B}\) is equivalent to \((\mathcal{G}, \mathcal{B})\)-measurability of the function \(f\). This can be seen if we define the measure \(\mu\) on \(\mathcal{G}\) by \(\mu = 0\) for the void set and \(\mu = \infty\) for all other sets in \(\mathcal{G}\).

2. That neither in (A) nor in (B) the assumption on \((Y, \mathcal{B})\) to be a Blackwell space is superfluous can be seen by the following simple example: \(X = Y = I\) (the unit interval), \(\mathcal{G}\) the Borel sets in \(I\), \(\mathcal{B}\) the \(\sigma\)-field generated by the Borel sets and one set \(V\) being nonmeasurable with respect to Lebesgue measure, and \(\mu\) the Lebesgue measure on \(\mathcal{G}\). If \(f\) is the identity on \(I\), then \(\text{gr}(f) \in \mathcal{G}_\mu \otimes \mathcal{B}\), but the function is not \((\mathcal{G}_\mu, \mathcal{B})\)-measurable, since \(f^{-1}(V) \not\subset \mathcal{G}_\mu\).

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References


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