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A GENERALIZATION OF A THEOREM OF S. N. BERNSTEIN[1]

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ABSTRACT. A rational approximation scheme is exhibited for a set Δ which consists of a finite union of compact subintervals of the real line. This rational approximation scheme provides a characterization of the analytic functions on Δ which generalizes S. N. Bernstein's characterization of the analytic functions on $[-1, 1]$.

1. **Bernstein's theorem.** Let E be a compact subset of the real line R. A complex-valued function f defined on E is said to be *analytic* on E if f is the restriction to E of a holomorphic function F defined on an open subset Ω of the complex plane $\mathbf{C}$ that contains E. Note that Ω need not be a region.

A theorem of S. N. Bernstein relates the analyticity of a function f defined on the interval $I = [-1, 1]$ to the rate by which f may be uniformly approximated by polynomials on I. In order to state Bernstein's theorem precisely, let P_n be the class of all polynomials of degree $\leqslant n$, let

$$\|g\|_I = \sup\{|g(x)|: x \in I\}$$

for any function g defined on I, and let

$$e(f, P_n, I) = \inf\{\|f - p\|_I: p \in P_n\} \qquad (n = 1, 2, 3, \ldots).$$

Let D_ρ be the open elliptical region defined by

$$D_\rho = \{z \in \mathbf{C}: |z + 1| + |z - 1| < \rho + 1/\rho\}, \tag{1}$$

for $0 < \rho < 1$.

THEOREM 1 (S. N. BERNSTEIN). *Let f be a complex-valued function defined on $I = [-1, 1]$. Then*

$$\limsup_{n\to\infty} \sqrt[n]{e(f, P_n, I)} \leqslant \rho, \tag{2}$$

with $0 < \rho < 1$, if and only if f is the restriction to I of a holomorphic function F defined on the region D_ρ. In particular,

$$\lim_{n\to\infty} \sqrt[n]{e(f, P_n, I)} = 0 \tag{3}$$

if and only if the holomorphic function F is entire.

PROOF. See [1, pp. 110–113].

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Suppose one replaces the interval I by a union of intervals

$$\Delta = \bigcup_{j=0}^{r} [a_j, b_j],$$

where $-\infty < a_0 < b_0 < a_1 < \cdots < b_r < +\infty$, and seeks a similar characterization of the analytic functions on Δ. If one thinks of polynomials as rational functions with poles at the point $\lambda_0 = \infty$, then one might try to generalize Bernstein's theorem to Δ by replacing the class of polynomials by a class of rational functions with poles at certain points $\lambda_1, \ldots, \lambda_r$ between each of the intervals $[a_0, b_0], [a_1, b_1], \ldots, [a_r, b_r]$, as well as at the point $\lambda_0 = \infty$. The difficulty in finding such a generalization is choosing the poles $\lambda_1, \ldots, \lambda_r$.

In this note the work of Rosenblum and Rovnyak on Cayley inner functions (see [3]) is used to develop a rational approximation scheme for Δ that corresponds to polynomial approximation on I and provides a characterization of the analytic functions on Δ analogous to that in Bernstein's theorem.

2. Let $\Delta = \bigcup_{j=0}^{r}[a_j, b_j]$, where $-\infty < a_0 < b_0 < a_1 < \cdots < b_r < +\infty$, and let ϕ be the rational function

$$\phi(z) = \left(b - a \prod_{j=0}^{r} \frac{b_j - z}{a_j - z}\right) \Big/ \left(1 - \prod_{j=0}^{r} \frac{b_j - z}{a_j - z}\right).$$

The poles of $\phi(z)$ are the solutions of the equation $1 = \prod_{j=0}^{r}(b_j - z)/(a_j - z)$, and hence are at the points $z = \infty$ and $z = \lambda_1, \ldots, \lambda_r$, where $b_{j-1} < \lambda_j < a_j$ $(j = 1, \ldots, r)$. Moreover,

$$\phi^{-1}([a, b]) = \Delta \quad \text{and} \quad \phi^{-1}([a, b]^c) = \Delta^c \setminus \{\lambda_1, \ldots, \lambda_r\}$$

(where $E^c = \{x \in R: x \not\in E\}$ whenever $E \subseteq R$), and ϕ is a strictly increasing mapping of each of the intervals $[a_0, b_0], [a_1, b_1], \ldots, [a_r, b_r]$ onto $[a, b]$. The rational function ϕ is an example of a *Cayley inner function* and is said to *adapt* Δ to $[a, b]$; see [3] for details.

Let $\lambda_0 = -\infty$, $\lambda_{r+1} = +\infty$, and put

$$\phi_j = \phi|(\lambda_j, \lambda_{j+1}) \qquad (j = 0, 1, \ldots, r).$$

Define the functions $k_0, k_1, \ldots, k_r$ on Δ by

$$k_0(t) = 1, \quad k_1(t) = (t - \lambda_1)^{-1}, \ldots, k_r(t) = (t - \lambda_r)^{-1}.$$

(We use t as a dummy variable for functions defined on Δ and x for functions defined on $[a, b]$.) If E is a compact subset of the real line R, then $C(E)$ is the Banach space of all complex-valued continuous functions on E with norm $\|f\|_E = \sup\{|f(x)|: x \in E\}$; $C(E)^{r+1}$ is the direct sum of $r + 1$ copies of $C(E)$ and is a Banach space under any of several equivalent norms.

LEMMA. *Define the linear operator* B: $C([a, b])^{r+1} \to C(\Delta)$ *by*

$$(Bg)(t) = \sum_{j=0}^{r} k_j(t)\, g_j(\phi(t)),$$

where $g = (g_0, g_1, \ldots, g_r)$. *Then* B *is bounded and has a bounded inverse defined by*

$$(B^{-1}f)(x) = A(x)^{-1} \cdot \begin{bmatrix} f\left(\phi_0^{-1}(x)\right) \\ \vdots \\ f\left(\phi_r^{-1}(x)\right) \end{bmatrix}, \qquad (f \in C(\Delta)),$$

where $A(x)$ *is the matrix*

$$A(x) = \left[a_{ij}(x)\right]_{i,j=0}^{r} = \left[k_j\left(\phi_i^{-1}(x)\right)\right]_{i,j=0}^{r}.$$

PROOF. If $g = (g_0, g_1, \ldots, g_r) \in C([a, b])^{r+1}$, then

$$\|Bg\|_\Delta \leqslant (r+1) \cdot K \cdot \max_j \|g_j \circ \phi\|_\Delta = (r+1) \cdot K \cdot \max_j \|g_j\|_{[a,b]},$$

where $K = \max_j \|k_j\|_\Delta < +\infty$. Hence B is a bounded operator.

If $f \in C(\Delta)$, then $f = Bg$ with $g = (g_0, g_1, \ldots, g_r)$ if and only if

$$f\left(\phi_i^{-1}(x)\right) = \sum_{j=0}^{r} k_j\left(\phi_i^{-1}(x)\right) g_j(x) \qquad (i = 0, 1, \ldots, r);$$

that is, if and only if

$$f\left(\phi_i^{-1}(x)\right) = \sum_{j=0}^{r} a_{ij}(x)\, g_j(x) \qquad (i = 0, 1, \ldots, r),$$

where $a_{ij}(x) = k_j(\phi_i^{-1}(x))$ for $i, j = 0, 1, \ldots, r$. The matrix $A(x) = [a_{ij}(x)]$ has the form

$$\begin{bmatrix} 1, & \dfrac{1}{\eta_0 - \lambda_1} & ,\ldots, & \dfrac{1}{\eta_0 - \lambda_r} \\ \cdots & & & \\ 1, & \dfrac{1}{\eta_r - \lambda_1} & ,\ldots, & \dfrac{1}{\eta_r - \lambda_r} \end{bmatrix},$$

where $\eta_i = \phi_i^{-1}(x)$. Since

$$\det\left[\frac{1}{\eta_i - \lambda_j}\right]_{i,j=0}^{r} = \frac{\prod_{r \geqslant i > j \geqslant 0}(\eta_i - \eta_j)(\lambda_j - \lambda_i)}{\prod_{i,j=0,\ldots,n}(\eta_i - \lambda_j)}$$

(see [2, part seven, Problem 3]), we deduce that

$$\det A(x) = \lim_{\lambda_0 \to \infty}\left(-\lambda_0 \det\left[\frac{1}{\eta_i - \lambda_j}\right]_{i,j=0}^{r}\right) \neq 0$$

for $a \leqslant x \leqslant b$. Thus $A(x)$ is invertible for $a \leqslant x \leqslant b$, and

$$A(x)^{-1} = [d_{ij}(x)], \qquad d_{ij}(x) = C_{ji}(x)/\det A(x),$$

where $C_{ij}(x)$ is the cofactor of $a_{ij}(x)$ in $A(x)$. It follows that if f is in $C(\Delta)$ and if

$$g(x) = \begin{bmatrix} g_0(x) \\ g_1(x) \\ \cdot \\ \vdots \\ g_r(x) \end{bmatrix} = A(x)^{-1} \cdot \begin{bmatrix} f(\phi_0^{-1}(x)) \\ f(\phi_1^{-1}(x)) \\ \cdot \\ \vdots \\ f(\phi_r^{-1}(x)) \end{bmatrix},$$

then $g \in C([a, b])^{r+1}$, $f = Bg$, and $\|g_i\|_{[a,b]} \leqslant M\|f\|_\Delta$ $(i = 0, 1, \ldots, r)$ for some finite constant M independent of f. This completes the proof.

Since the matrix $A(x) = [a_{ij}(x)]$ has analytic entries, it follows that, if $f = Bg$ and $g = (g_0, g_1, \ldots, g_r)$, then f is analytic on Δ if and only if g_0, $g_1, \ldots, g_r$ are analytic on $[a, b]$. For $n = 1, 2, 3, \ldots$, let K_n be the complex vector space spanned by the functions

$$1, t, \ldots, t^n; \ \frac{1}{t - \lambda_1}, \ldots, \frac{1}{(t - \lambda_1)^{n+1}}; \ \ldots; \ \frac{1}{t - \lambda_r}, \ldots, \frac{1}{(t - \lambda_r)^{n+1}}.$$

Then K_n is the image under B of all $(r + 1)$-tuples of polynomials of degree $\leqslant n$; see [3, Lemma 2.4]. The class K_n will replace the class of polynomials P_n in our generalization of Bernstein's theorem to Δ.

3. **A generalization of Bernstein's theorem.** For simplicity, we let $[a, b]$ be the interval $I = [-1, 1]$. For each complex-valued function f defined on Δ, put

$$e(f, K_n, \Delta) = \inf\{\|f - k\|_\Delta : k \in K_n\} \quad \text{for } n = 1, 2, 3, \ldots.$$

THEOREM 2. *A complex-valued function f defined on Δ is analytic on Δ if and only if*

$$\limsup_{n\to\infty} \sqrt[n]{e(f, K_n, \Delta)} < 1, \tag{4}$$

and

$$\lim_{n\to\infty} \sqrt[n]{e(f, K_n, \Delta)} = 0 \tag{5}$$

if and only if f has a holomorphic extension F to the region $\Omega_\phi = \{z \in \mathbf{C}$: $z \neq \lambda_1, \ldots, z \neq \lambda_r\}$. There exists a real number ρ that depends only on ϕ, with $0 < \rho < 1$, *such that f has a holomorphic extension F to a region if*

$$\limsup_{n\to\infty} \sqrt[n]{e(f, K_n, \Delta)} < \rho. \tag{6}$$

PROOF. Let f be a function on Δ with

$$\limsup_{n\to\infty} \sqrt[n]{e(f, K_n, \Delta)} < \sigma < 1.$$

Then there exists an integer N such that $e(f, K_n, \Delta) < \sigma^n$ if $n \geqslant N$. Hence f is the uniform limit of continuous functions on Δ and so $f \in C(\Delta)$. By the Lemma, there exist continuous functions $g_0, g_1, \ldots, g_r$ on I such that $f = Bg$, where $g = (g_0, g_1, \ldots, g_r)$. For each $n \geqslant N$, there is a function p_n in K_n with $\|f - p_n\|_\Delta < \sigma^n$ and there exist polynomials $p_{n0}, p_{n1}, \ldots, p_{nr}$ such that $p_n = B(p_{n0}, p_{n1}, \ldots, p_{nr})$. By the Lemma, there is a finite constant M such that

$$\|g_j - p_{nj}\|_I \leqslant M\|f - p_n\|_\Delta < M\sigma^n \qquad (n \geqslant N; j = 0, 1, \ldots, r).$$

Thus

$$\limsup_{n\to\infty} \sqrt[n]{e(g_j, P_n, I)} \leqslant \sigma < 1 \qquad (j = 0, 1, \ldots, r)$$

and so the functions $g_0, g_1, \ldots, g_r$ are analytic on I by Bernstein's theorem. Thus $f = Bg$ is analytic on Δ.

Moreover, if (5) holds for f, then

$$\limsup_{n\to\infty} \sqrt[n]{e(g_j, P_n, I)} = 0 \qquad (j = 0, 1, \ldots, r)$$

and so each g_j has a holomorphic extension to an entire function G_j by Bernstein's theorem. Hence

$$F(z) = \sum_{j=0}^{r} k_j(z) G_j(\phi(z)) \qquad (z \in \Omega_\phi)$$

is a holomorphic extension of f to Ω_ϕ.

A simple connectedness argument shows there exists a real number ρ in (0, 1) such that $\phi^{-1}(D_\rho)$ is a *region*, where D_ρ is defined by (1). Suppose (6) holds for f. Put $\sigma = \rho$ in the above argument; then Bernstein's theorem implies that each g_j has a holomorphic extension G_j to D_ρ. If $F(z) = \sum_j k_j(z) G_j(\phi(z))$ for z in $\phi^{-1}(D_\rho)$, then F is a holomorphic extension of f to the region $\phi^{-1}(D_\rho)$.

Conversely, if f is analytic on Δ, then there exist analytic functions $g_0, g_1, \ldots, g_r$ on I such that $f = Bg$, where $g = (g_0, g_1, \ldots, g_r)$. By Bernstein's theorem (2) and the Lemma, inequality (4) holds for f.

Suppose, in addition, that f has a holomorphic extension F to Ω_ϕ. Cauchy's integral formula shows that

$$F = F_0 + F_1 + \cdots + F_r, \tag{7}$$

where F_0 is entire and F_j is holomorphic in $\mathbf{C}_\infty \setminus \{\lambda_j\}$ (where $\mathbf{C}_\infty$ is the Riemann sphere $\mathbf{C} \cup \{\infty\}$), for $j = 1, \ldots, r$. By Bernstein's theorem (3),

$$\lim_{n\to\infty} \sqrt[n]{e(F_0, P_n, [a_0, b_r])} = 0.$$

Since $P_n \subseteq K_n$ and $\Delta \subseteq [a_0, b_r]$,

$$\lim_{n\to\infty} \sqrt[n]{e(F_0, K_n, \Delta)} = 0. \tag{8}$$

To establish the same result for F_j, we put $\zeta_j(z) = (z - \lambda_j)^{-1}$. Then ζ_j maps $\mathbf{C}_\infty \setminus \{\lambda_j\}$ conformally onto $\mathbf{C}$ and maps $[-\infty, b_{j-1}] \cup [a_j, +\infty]$ onto a compact interval I_j of the real line. Also, $F_j = G_j \circ \zeta_j$, where G_j is entire. By Bernstein's theorem (3),

$$\lim_{n\to\infty} \sqrt[n]{e(G_j, P_n, I_j)} = 0. \tag{9}$$

Let $P_n(\lambda_j) = \{p \circ \zeta_j : p \in P_n\}$ for $n = 1, 2, 3, \ldots$; then $P_n(\lambda_j) \subseteq K_n$. Since

$$\|G_j - p\|_{I_j} \geqslant \|G_j \circ \zeta_j - p \circ \zeta_j\|_\Delta = \|F_j - p \circ \zeta_j\|_\Delta \qquad \left(p \in \bigcup_{n=1}^{\infty} P_n\right),$$

equation (9) implies that

$$\lim_{n\to\infty} \sqrt[n]{e(F_j, P_n(\lambda_j), \Delta)} = 0$$

and hence that

$$\lim_{n\to\infty} \sqrt[n]{e(F_j, K_n, \Delta)} = 0 \qquad (j = 1, \ldots, r). \tag{10}$$

By (7), (8) and (10), equation (5) holds.

This completes the proof.

References

1. S. N. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Gauthiers-Villars, Paris, 1926.

2. G. Pólya and G. Szegö, *Problems and theorems in analysis*. Vol. II, Springer-Verlag, Berlin and New York, 1976.

3. M. Rosenblum and J. Rovnyak, *Cayley inner functions and best approximations*, J. Approximation Theory **17** (1976), 241–253.

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