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A NOTE ON PSEUDOCOMPACT SPACES AND k_R-SPACES

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ABSTRACT. Utilizing the Stone-Čech compactification of an uncountable discrete space, we construct a pseudocompact space X which belongs to Frolík's class $\mathfrak{P}^*$ but $k_R X$ is not pseudocompact.

All spaces in this paper will be completely regular Hausdorff. Recall that a space X is called a k_R-space provided each real-valued function on X is continuous if its restriction to every compact subset of X is continuous, and that associated with each space X there is a unique k_R-space $k_R X$ having the same underlying set and the same compact sets as X.[1] Let $\mathfrak{K}$ be the class of spaces X such that $k_R X$ is pseudocompact. Let $\mathfrak{P}^*$ be the class of pseudocompact spaces X with the property: Each infinite collection of disjoint open sets has an infinite subcollection, each of which meets some fixed compact set [2].

N. Noble [3] showed that $\mathfrak{K} \subset \mathfrak{P}^*$ and that $\mathfrak{P}^*$ is closed under arbitrary products. In [4] he proved $\mathfrak{K}$ is also closed under arbitrary products. It was not known, however, whether the two classes coincide or not. The purpose of this note is to show that they differ, i.e., *$\mathfrak{K}$ is properly contained in* $\mathfrak{P}^*$.[2]

For a space X, βX and X^* denote the Stone-Čech compactification of X and its remainder, respectively. For D the discrete set of power $\aleph_1$, let Ω be the subspace of $D^* = \beta D \setminus D$ consisting of all the elements in the closure (in βD) of some countable subset of D. Let A be a countably infinite, discrete subset of $D^* \setminus \Omega$. Put $X = \Omega \cup A \subset D^*$. Henceforth X denotes this subspace of D^*. We will show that $X \in \mathfrak{P}^* \setminus \mathfrak{K}$.

ASSERTION 1. *X belongs to* $\mathfrak{P}^*$.

PROOF. Since every countable subset of Ω has compact closure in Ω, Ω belongs to $\mathfrak{P}^*$. Since Ω is dense in X, X also belongs to $\mathfrak{P}^*$.

The next property of D^* is the key to prove that X is not in $\mathfrak{K}$.

LEMMA 1. *Let F be a noncompact closed subset of Ω. Then $\mathrm{cl}_{D^*} F \setminus \Omega$ is infinite. In fact, its cardinal is at least* $\exp \exp \aleph_1$.

PROOF. Let F be a noncompact closed set in Ω. Put $uD = D^* \setminus \Omega$. It is well known that the cardinal of uD is $\exp \exp \aleph_1$ (cf. [5, Theorem 5.13]). Hence,

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[1] $k_R X$ has the coarsest topology making continuous each real-valued function on X whose restriction to compact subsets is continuous.

[2] Independent of the author, this fact was also proved by J. L. Blasco [1] (with a different example).

we need only show that $\mathrm{cl}_{D^*}F \setminus \Omega$ contains a copy of uD.

Let $x_0 \in F$. Then $x_0 \in \mathrm{cl}_{\beta D} N_0$ for some countable set N_0 in D. Let us denote by ω_1 the first uncountable ordinal. Suppose $\alpha < \omega_1$ and that we have chosen a subset $\{x_\gamma\}_{\gamma<\alpha}$ of F and a family $\{N_\gamma\}_{\gamma<\alpha}$ of disjoint countable subsets of D with $x_\gamma \in \mathrm{cl}_{\beta D} N_\gamma$. Since $\bigcup_{\gamma<\alpha} N_\gamma$ is countable, its closure in βD is a compact set contained in Ω; hence $F \setminus \mathrm{cl}_{\beta D} \bigcup_{\gamma<\alpha} N_\gamma$ is nonempty because F is not compact. Pick a point x from the nonempty set. Since every point of Ω has a neighborhood (in βD) which is a closure of some countable set of D, there exists a countable set N in D, disjoint from $\bigcup_{\gamma<\alpha} N_\gamma$, with $x \in \mathrm{cl}_{\beta D} N$. Put $x_\alpha = x$ and $N_\alpha = N$.

Thus, by induction, we get a subset $\{x_\alpha\}_{\alpha<\omega_1}$ of F and a family $\{N_\alpha\}_{\alpha<\omega_1}$ of disjoint countable subsets of D such that $x_\alpha \in \mathrm{cl}_{\beta D} N_\alpha$. Put $F_1 = \{x_\alpha\}_{\alpha<\omega_1}$. Clearly F_1 is a copy of D. We will show next that F_1 is C^*-embedded in βD. Let f be a bounded real-valued function on F_1. Define a function f_D on D by $f_D(N_\alpha) = f(x_\alpha)$ and $f_D(D \setminus \bigcup_{\alpha<\omega_1} N_\alpha) = 0$. Then the Stone extension of f_D over βD is an extension of f. Hence F_1 is C^*-embedded in βD and this implies $\beta F_1 = \mathrm{cl}_{D^*} F_1 \subset \mathrm{cl}_{D^*} F$. Thus we have $\beta F_1 \setminus \Omega \subset \mathrm{cl}_{D^*} F \setminus \Omega$. Now it is easy to see that $\beta F_1 \setminus \Omega$ is a copy of uD. This completes the proof.

LEMMA 2. *Every compact subset K of $X = \Omega \cup A$ is a topological sum $K = K_1 \oplus K_2$ of a compact set K_1 in Ω and a finite set K_2 in A.*

PROOF. Let K be a compact set in X. Put $K_1 = K \cap \Omega$ and $K_2 = K \cap A$. Note that A is a closed subset of X because Ω is locally compact. Therefore K_2 is a compact set in the discrete space A, i.e., K_2 is finite. Since K is compact, $\mathrm{cl}_{D^*} K_1 \setminus \Omega$ is contained in the finite set K_2. Hence, by Lemma 1, K_1 is compact.

ASSERTION 2. *X does not belong to $\mathfrak{K}$, i.e., $k_R X$ is not pseudocompact.*

PROOF. Let $a \in A$ and let f_a be a real-valued function on X such that $f_a(a) = 1$ and $f_a(x) = 0$ for any $x \neq a$. Then, by Lemma 2, f_a is continuous on every compact subset of X. This implies that each point of A is isolated in $k_R X$. Since Ω is locally compact, we have $k_R X = \Omega \oplus A$. Now it is clear that $k_R X$ is not pseudocompact.

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