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ON THE NORMAL SPECTRUM OF A SUBNORMAL OPERATOR

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ABSTRACT. In this note we present a new characterization for subnormality which is purely C^*-algebraic. We also establish an intrinsic characterization of the normal spectrum for a subnormal operator, which enables us to prove that $\sigma_\perp(\pi(S)) \subseteq \sigma_\perp(S)$ for any *-representation π.

An operator A on a Hilbert space $\mathcal{H}$ is called subnormal if there exists a normal operator N on a larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $A = N|\mathcal{H}$. P. R. Halmos [4] introduced subnormal operators and showed that there is always an essentially unique minimal normal extension.

Following M. B. Abrahamse and R. G. Douglas [1] we define the normal spectrum $\sigma_\perp(S)$ of a subnormal operator S to be the spectrum of the minimal normal extension of S. J. Bram [2] continued the study of subnormal operators and showed that subnormality is preserved under *-representations and that $\partial\sigma(S) \subseteq \sigma_\perp(S) \subseteq \sigma(S)$. In this note we use Bram's results to give a new characterization for subnormality that is purely C^*-algebraic. It easily follows that subnormality is preserved by *-representations. We also give an intrinsic characterization of the normal spectrum $\sigma_\perp(S)$, from which it easily follows that $\sigma_\perp(\pi(S)) \subseteq \sigma_\perp(S)$ for any *-representation π of $C^*(S)$. This answers problem 2 in [1] in the affirmative. For any operator T, $C^*(T)$ will denote the C^*-algebra generated by T.

PROPOSITION 1. *An operator A on a Hilbert space $\mathcal{H}$ is subnormal if and only if $\sum_{i,j=0}^n B_i^* A^{*j} A^i B_j \geqslant 0$ for every finite set*

$$B_0, B_1, B_2, \ldots, B_n \in C^*(A).$$

PROOF. It follows from [4] and Theorem 1 in [2] that A is subnormal if and only if $\sum_{i,j=0}^n (A^i f_j, A^j f_i) \geqslant 0$ for every finite set of vectors $f_0, f_1, \ldots, f_n \in \mathcal{H}$. For $B_0, B_1, \ldots, B_n \in C^*(A)$ and $x \in \mathcal{H}$, let $f_i = B_i x$. We see that if A is subnormal, then

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$$0 \leqslant \sum_{i,j=0}^{n} \left(A^i f_j, A^j f_i\right) = \sum_{i,j=0}^{n} \left(A^i B_j x, A^j B_i x\right)$$
$$= \sum_{i,j=0}^{n} \left(B_i^* A^{*j} A^i B_j x, x\right) = \left(\left\{ \sum_{i,j=0}^{n} B_i^* A^{*j} A^i B_j \right\} x, x\right)$$

for every $x \in \mathcal{H}$. Hence $0 \leqslant \Sigma_{i,j=0}^{n} B_i^* A^{*j} A^i B_j$. Conversely, let $f_0, f_1, \ldots, f_n \in \mathcal{H}$ be given. It clearly suffices to assume that all the f_j's lie in a subspace $\mathfrak{M}$ cyclic for $C^*(A)$, with cyclic vector x_0. Then there exist $B_{kj} \in C^*(A)$ with $f_j = \lim_{k\to\infty} B_{kj} x_0$. Hence

$$\sum_{i,j=0}^{n} \left(A^i f_j, A^j f_i\right) = \lim_{k\to\infty} \left(\sum_{i,j=0}^{n} \left(A^i B_{kj} x_0, A^j B_{ki} x_0\right)\right)$$
$$= \lim_{k\to\infty} \left(\sum_{i,j=0}^{n} B_{ki}^* A^{*j} A^i B_{kj} x_0, x_0\right) \geqslant 0$$

and A is subnormal.

COROLLARY 1 (BRAM). *If S is subnormal and π is a *-representation of $C^*(S)$ then $\pi(S)$ is also subnormal.*

We now give an intrinsic characterization of the spectrum $\sigma_\perp(S)$ of the minimal normal extension N of a subnormal operator S. Write $\rho_\perp(S) = \mathbf{C} \setminus \sigma_\perp(S)$.

PROPOSITION 2. *If S is subnormal then $\lambda \in \rho_\perp(S)$ if and only if there exists $\alpha > 0$ such that*

$$\sum_{i,j=0}^{n} B_i^* S^{*j} (S-\lambda)^* (S-\lambda) S^i B_j \geqslant \alpha \sum_{i,j=0}^{n} B_i^* S^{*j} S^i B_j \tag{1}$$

for every finite set $B_0, B_1, \ldots, B_n \in C^*(S)$.

PROOF. Recall that for a normal operator the spectrum is equal to the approximate point spectrum. Thus $\lambda \in \rho_\perp(S) \Leftrightarrow \lambda \in \rho(N) \Leftrightarrow N - \lambda$ is bounded below $\Leftrightarrow$ there exists $\alpha > 0$ with $\|(N-\lambda)y\|^2 \geqslant \alpha \|y\|^2$ for all $y \in \mathcal{K}$. Since N is the minimal normal extension of S, $\mathcal{K}$ is the smallest reducing subspace for N containing $\mathcal{H}$. Thus $\mathcal{K} = \operatorname{span}\{N^{*i} x_0 \colon x_0 \in \mathcal{H}, i \geqslant 0\}$; thus $\lambda \in \rho_\perp(S) \Leftrightarrow$ there exists $\alpha > 0$ with $\|(N-\lambda)\Sigma_{i=0}^n N^{*i} f_i\|^2 \geqslant \alpha \|\Sigma_{i=0}^n N^{*i} f_i\|^2$, for every finite set $f_0, f_1, f_2, \ldots, f_n \in \mathcal{H}$. We may rewrite this last inequality as

$$\left((N-\lambda) \sum_{j=0}^{n} N^{*j} f_j, (N-\lambda) \sum_{i=0}^{n} N^{*i} f_i\right) \geqslant \alpha \left(\sum_{j=0}^{n} N^{*j} f_j, \sum_{i=0}^{n} N^{*i} f_i\right)$$

or (since N is normal, N and N^* commute)

$$\sum_{i,j=0}^{n} \left((N-\lambda) N^i f_j, (N-\lambda) N^j f_i\right) \geqslant \alpha \sum_{i,j=0}^{n} \left(N^i f_j, N^j f_i\right)$$

or (since $S = N|\mathcal{H}$)

$$(2) \qquad \sum_{i,j=0}^{n} \left((S-\lambda)S^i f_j, (S-\lambda)S^j f_i\right) \geq \alpha \sum_{i,j=0}^{n} \left(S^i f_j, S^j f_i\right).$$

Thus $\lambda \in \rho_\perp(S)$ if and only if (2) holds for every finite set $f_0, f_1, \ldots, f_n \in \mathcal{H}$. If we let $f_i = B_i x$ for $B_i \in C^*(S)$ and $x \in \mathcal{H}$ we obtain

$$\left(\sum_{i,j=0}^{n} \left(B_i^* S^{*j}(S-\lambda)^*(S-\lambda)S^i B_j\right)x, x\right) \geq \alpha\left(\sum_{i,j=0}^{n} B_i^* S^{*j} S^i B_j x, x\right)$$

which is just (1). To see that (1) implies (2), it again suffices to assume that all the f_i lie in a subspace cyclic for $C^*(S)$. The proof then follows precisely as in the proof of Proposition 1.

COROLLARY 2. *If S is subnormal and π is a *-representation of $C^*(S)$ then $\sigma_\perp(\pi(S)) \subseteq \sigma_\perp(S)$.*

PROOF. It follows immediately from Proposition 2 that $\rho_\perp(S) \subseteq \rho_\perp(\pi(S))$. The conclusion now follows by taking complements.

REMARK 1. It is easily seen that Propositions 1 and 2 remain valid when the condition that $B_i \in C^*(A)$ is replaced by either the condition $B_i \in \mathcal{B}(\mathcal{H})$ or by the condition $B_i \in \mathcal{P}(A, A^*) =$ the set of all noncommutative polynomials in A and A^*. In fact, one need only require that $B_i \in \mathcal{B}(A)$ where $\mathcal{B}(A)$ is any subset such that any cyclic subspace for $C^*(A)$ is also cyclic for $\mathcal{B}(A)$. For example, if A itself is cyclic one need only require $B_i \in \mathcal{P}(A)$, or if A^* is cyclic one need only require that $B_i \in \mathcal{P}(A^*)$.

REMARK 2. It follows from Corollary 2 that if two subnormal operators are algebraically equivalent (i.e., there exists a faithful representation π of $C^*(S_1)$ onto $C^*(S_2)$ with $\pi(S_1) = S_2$), then their minimal normal extensions are also algebraically equivalent.

REMARK 3. The normal spectrum $\sigma_\perp(S)$ behaves well under direct sums ($\sigma_\perp(S_1 \oplus S_2) = \sigma_\perp(S_1) \cup \sigma_\perp(S_2)$), but unlike the spectrum it is not upper semicontinuous. Consider the weighed shifts S_n with weights $\sqrt{n/(n+1)}$, $\sqrt{(n+1)/(n+2)}$, $\sqrt{(n+2)/(n+3)}, \ldots$ respectively. Then (see [3]) each S_n is subnormal with $\sigma_\perp(S_n) =$ closed unit disk, and S_n converges uniformly to the unilateral shift U_+ with $\sigma_\perp(U_+) =$ unit circle.

REMARK 4. Proposition 1 enables one to give C^*-algebraic proofs for certain properties of subnormal operators, the proofs of which normally involve the normal extension. For example, if A is subnormal then A is hyponormal. (PROOF. Take $n = 1$ and $B_0 = -A^*$, $B_1 = I$.) If A is an invertible subnormal then A^{-1} is subnormal. (PROOF. For $D_0, D_1, \ldots, D_n \in C^*(A^{-1}) = C^*(A)$ set $B_i = A^{-n}D_{n-i}$; then

$$0 \leq \sum_{i,j=0}^{n} B_i^* A^{*j} A^i B_j = \sum_{i,j=0}^{n} D_{n-i}^* A^{*-n} A^{*j} A^i A^{-n} D_{n-j}$$

$$= \sum_{i,j=0}^{n} D_{n-i}^* (A^{*-1})^{n-j} (A^{-1})^{n-i} D_{n-j}$$

and the obvious change of variables yields the desired result.)

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