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ON CONTINUITY OF FIXED POINTS OF COLLECTIVELY CONDENSING MAPS

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ABSTRACT. In this paper, we prove, in two parts, the following claim. Let X be a Banach space and Λ an arbitrary topological space. Suppose that $T\colon \Lambda \times X \to X$ is collectively condensing; then the fixed point set $S(\lambda,y)$ has closed graph if and only if T is continuous in both λ and y.

Zvi Artstein [1] proved

THEOREM 0. *Suppose X is a Banach space and Λ a metric space. Let $T\colon \Lambda \times X \to X$ be collectively condensing, i.e., for every $B \subset X$,*

$$\chi\left(\bigcup_{\lambda \in \Lambda} T(\lambda, B)\right) \leqslant \chi(B),$$

where equality implies $\chi(B) = 0$ and χ is the Kuratowski measure of noncompactness, defined as $\chi(B) = \inf\{d \mid B$ can be covered by a finite number of subsets of diameter $< d\}$. Let $S(\lambda,y) = \{x \in X \mid x = T(\lambda,x) + y\}$. Then, $S(\lambda,y)$ is upper-semicontinuous if and only if $T(\lambda,x)$ is continuous in λ and x simultaneously.

At the end of his paper, he posed an open question. Is Theorem 0 still true if Λ is a general topological space? Upon investigation we find that $S(\lambda,y)$ has closed graph if and only if $T(\lambda,x)$ is continuous in both λ and x.

The following definitions are intended to refresh the memory of the readers.

DEFINITION 0.1. Let E_1 be a topological space, and Λ a subset of a topological space E_2. We say a multifunction $F\colon \Lambda \to E_1$ is *upper-semicontinuous* at $\lambda_0 \in \Lambda$ if for each open set G in E_1 containing $F(\lambda_0)$, there exists an open neighborhood $U(\lambda_0)$ of λ_0 in E_2 such that $F(U(\lambda_0) \cap \Lambda) \subset G$.

DEFINITION 0.2. F is *upper-semicontinuous on* Λ if it is upper-semicontinuous at each $\lambda_0 \in \Lambda$.

To prove our main theorem, we need the following tools.

THEOREM 0.3 [2, THEOREM 11.5]. *A net has y as a cluster point iff it has a subnet which converges to y.*

THEOREM 0.4 [2, THEOREM 11.8]. *Let $f\colon X \to Y$. Then f is continuous at $x_0 \in X$ iff whenever $x_\lambda \to x_0$ in X, then $f(x_\lambda) \to f(x_0)$ in Y.*

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THEOREM 0.5 [2, THEOREM 11Dc]. *If every subnet of a net* (x_λ) *has a subnet converging to* x, *then* (x_λ) *converges to* x.

The following are our main theorems.

THEOREM 1. *Let* X *be a Banach space, and* Λ *a topological space. Let* $T\colon \Lambda \times X \to X$ *be collectively condensing. Suppose that* S *is upper-semicontinuous; then* T *is continuous in* λ *and* x.

PROOF. We will denote $T(\lambda, \cdot)$ by T_λ. Let $(\lambda_\alpha, x_\alpha)$ be a net in $\Lambda \times X$ converging to (λ_0, x_0). We will show that there exists a subnet $(T_{\lambda_{\alpha_\omega}} x_{\alpha_\omega})$ converging to $(T_{\lambda_0} x_0)$. Let $r_n = 1/2^n$, $n = 1, 2, \ldots$. Let $\mathfrak{T}_1 = \{x_\alpha \mid \|x_\alpha - x_0\| \leqslant r_1\}$. We claim $\mathfrak{T}_1$ is a subnet of (x_α). Indeed, a routine argument shows that the set $A_1 = \{\alpha \mid x_\alpha \in \mathfrak{T}_1\}$ is a directed set, cofinal with the given directed set. Consequently, $\{(\lambda_\alpha, x_\alpha) \mid \alpha \in A_1\}$ is a subnet, whereas $\mathfrak{T}_1$ and $\{T_{\lambda_\alpha} x_\alpha \mid \alpha \in A_1\}$ are subnets of (x_α) and $(T_{\lambda_\alpha} x_\alpha)$, respectively.

Let $x_\alpha, x_\beta \in \mathfrak{T}_1$. Then $\|x_\alpha - x_\beta\| \leqslant \|x_\alpha - x_0\| + \|x_\beta - x_0\| \leqslant 2r_1$. Hence, $\operatorname{diam}(\mathfrak{T}_1) \leqslant 2r_1$. By collective condensingness, $\chi(\{T_{\lambda_\alpha}(x_\alpha) \mid \alpha \in A_1\}) < \chi(\mathfrak{T}_1) \leqslant 2r_1$. Let $d_1 = \chi(\{T_{\lambda_\alpha}(x_\alpha) \mid \alpha \in A_1\})$. Choose $\varepsilon_1 > 0$ so that $d_1 + \varepsilon_1 < 2r_1$. Then, there exists a finite cover: $S_1^1, S_2^1, \ldots, S_{k(1)}^1$ of $\mathbb{S}_1 = \{T_{\lambda_\alpha}(x_\alpha) \mid \alpha \in A_1\}$. We claim that there is a set S_j^1, $1 \leqslant j \leqslant k(1)$, which contains a subnet of $\mathbb{S}_1$. The following is the proof of this claim.[1] Let $\mathbb{S} = \bigcup_{i=2}^{k(1)} S_i^1$, $A_{\mathbb{S}} = \{\alpha \in A_1 \mid T_{\lambda_\alpha}(x_\alpha) \in \mathbb{S}\}$, and $A_{S_i^1} = \{\alpha \in A_1 \mid T_{\lambda_\alpha}(x_\alpha) \in S_i^1\}$. Then $\{S_1^1, \mathbb{S}\}$ is a cover of $\mathbb{S}_1$, and $A_{S_1^1} \cup A_{\mathbb{S}} = A_1$. Suppose that S_1^1 does not contain a subnet of $\mathbb{S}_1$. Then there exists $\alpha \in A_1$ such that $d \geqslant \alpha$ implies that $d \notin A_{S_1^1}$. Let

$$D = \{d \in A_1 \mid d \geqslant \alpha \text{ for some } \alpha \in A_1, \text{ and if } \bar{d} \geqslant d, \text{ then } \bar{d} \notin A_S\}.$$

We will show that D is a directed set cofinal with A_1. Clearly, $D \subseteq A_{\mathbb{S}}$. We need only show that if $\alpha \in A$, then there exists a $d \in D$ such that $d \geqslant \alpha$. Since A_1 is a directed set, it follows that there exists a $\bar{d} \in A_1$ such that $\bar{d} \geqslant \alpha$. Let $\bar{D} = \{\bar{d} \in A_1 \mid \bar{d} \geqslant \alpha\}$. Clearly, $\bar{D}$ is a directed set cofinal with A_1. Since S_1^1 does not contain a subnet of $\mathbb{S}_1$, we can find $d^* \in A_1$, $d^* \geqslant \alpha$ such that $\bar{d} \geqslant d^*$ implies $\bar{d} \notin A_{S_1^1}$. Hence, $d^* \in D$. Thus, D is a directed set cofinal with A_1. This means that if S_1^1 does not contain a subnet, then $\mathbb{S}$ must contain one. By repeating the argument a finite number of times we see that there exists S_j^1, $1 \leqslant j \leqslant k(1)$, which contains a subnet of $\mathbb{S}_1$. Let $S_1 = S_j^1$. Let $\mathfrak{N}_1$ denote the subset contained in S_1. Similarly, we can define $\mathfrak{N}_2, \mathfrak{N}_3, \ldots$. Clearly, $\mathfrak{N}_1 \supset \mathfrak{N}_2 \supset \cdots$, where $\mathfrak{N}_n$ is of diameter $2r_n = 2/2^n = 1/2^{n-1}$, $n = 1, 2, \ldots$. Thus $\bigcap_{n=1}^{\infty} \overline{\mathfrak{N}}_n = x$ for some $x \in X$. Let $B_{r_n}(x)$ be an open ball in X with center at x and radius r_n. Then $B_{r_n}(x)$ contains a subnet of $(T_{\lambda_\alpha}(x_\alpha))$. This can be seen as follows. Consider $B_{r_{n+1}}(x)$. By construction, there exists $T_{\lambda_{\alpha_{n+2}}}(x_{\alpha_{n+2}}) \in \mathfrak{N}_{n+2}$ such that $T_{\lambda_{\alpha_{n+2}}}(x_{\alpha_{n+2}}) \in B_{r_{n+1}}(x)$. Thus, for each $T_{\lambda_\alpha} x_\alpha \in \mathfrak{N}_{n+2}$,

[1] The proof of this claim is essentially due to Jack Porter.

$$\|T_{\lambda_\alpha}(x_\alpha) - x\| \leqslant \|T_{\lambda_\alpha}(x_\alpha) - T_{\lambda_{\alpha_{n+2}}}(x_{\alpha_{n+2}})\| + \|T_{\lambda_{\alpha_{n+2}}}(x_{\alpha_{n+2}}) - x\| < 2r_{n+1} = r_n,$$

i.e., $\mathfrak{N}_{n+2} \subset B_{r_n}(x)$. This means, in other words, x is a cluster point of $\mathfrak{N}_1$. Hence, by Theorem 0.3, there is a subnet $\mathfrak{N}$ of $\mathfrak{N}_1$ which converges to x. Let $\mathfrak{N} = (T_{\lambda_{\alpha_\omega}}(x_{\alpha_\omega}))$. Then, clearly, $((\lambda_{\alpha_\omega}, x_{\alpha_\omega})) \to (\lambda_0, x_0)$. Let $y = x_0 - x$, and for each α_ω, let $y_{\alpha_\omega} = x_{\alpha_\omega} - T_{\lambda_{\alpha_\omega}}(x_{\alpha_\omega})$. Then, $(y_{\alpha_\omega}) \to y$, for some $y \in X$. By hypothesis, $x_0 \in S(\lambda_0, y)$, i.e., $x_0 = T_{\lambda_0}(x_0) - y$. Thus, $x = T_{\lambda_0}(x_0)$. Hence, $(T_{\lambda_{\alpha_\omega}}(x_{\alpha_\omega})) \to (T_{\lambda_0}(x_0))$. This actually shows that for each subnet of $(T_{\lambda_\alpha} x_\alpha)$, there is a subsubnet converging to $T_{\lambda_0}(x_0)$. By Theorem 0.5, $(T_{\lambda_\alpha}(x_\alpha)) \to T_{\lambda_0}(x_0)$. By Theorem 0.4, T is continuous in (λ, x). This completes the proof of Theorem 1.

REMARK 1. The above theorem is still true if S has closed graph instead. The proof is essentially the same.

THEOREM 2. *Let X be a Banach space, and Λ a topological space. Let $T\colon \Lambda \times X \to X$ be continuous. For each $(\lambda, y) \in \Lambda \times X$, let $S(\lambda, y) = \{x \in X \mid x = T(\lambda, x) + y\}$. Then S has closed graph.*

PROOF. Let $(\lambda_\alpha, y_\alpha)$ be a net converging to (λ_0, y_0). For each α let $x_\alpha \in S(\lambda_\alpha, y_\alpha)$, i.e., $x_\alpha = T(\lambda_\alpha, x_\alpha) + y_\alpha$. We will show that if $\bar{x}$ is a cluster point of (x_α), then $\bar{x} \in S(\lambda_0, y_0)$. Let $(\lambda_{\alpha_\omega})$ be a subnet of (x_α) converging to $\bar{x}$. Then, clearly, $(\lambda_{\alpha_\omega}) \to \lambda_0$, $(y_{\alpha_\omega}) \to y_0$. Hence, $(T(\lambda_{\alpha_\omega}, x_{\alpha_\omega})) = (x_{\alpha_\omega} - y_{\alpha_\omega}) \to \bar{x} - y_0$. By continuity, $(T(\lambda_{\alpha_\omega}, x_{\alpha_\omega})) \to T(\lambda_0, \bar{x})$. By Hausdorff property, $T(\lambda_0, \bar{x}) = \bar{x} - y_0$, i.e., $\bar{x} = T(\lambda_0, \bar{x}) + y_0$, or $\bar{x} \in S(\lambda_0, y_0)$. Hence, S has closed graph.

REMARK 2. Fred S. Van Vleck proved that if X is a Euclidean space, then the result of Theorem 0 is still valid no matter what space Λ is. However, if X is an arbitrary Banach space, our conjecture is that S need not be upper-semicontinuous even though T is continuous and collectively condensing.

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