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ON COMMUTANTS OF REDUCTIVE ALGEBRAS

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ABSTRACT. It is proved, for certain operators T, if $\mathcal{A}$ is a reductive algebra and $T \in \mathcal{A}'$, then $T^* \in \mathcal{A}'$.

1. **Introduction.** An algebra of bounded linear operators on a Hilbert space is *reductive* if it is weakly closed, contains the identity and has the property that each of its invariant subspaces is reducing. In [6], Rosenthal considered the following property an operator T may have in relation to reductive algebras:

(P) If $\mathcal{A}$ is any reductive algebra and $T \in \mathcal{A}'$, then $T^* \in \mathcal{A}'$.

Rosenthal [6] proved that (P) holds for compact operators. In this note we give a different proof of this result. Also we prove that (P) holds for operators quasi-similar to normal operators.

The following lemma from Feintuch and Rosenthal [1] is useful throughout this note:

LEMMA 1. *If* $T^2 = 0$, *then* T *has property* (P).

COROLLARY [5, p. 169]. *Property* (P) *holds for algebraic operators.*

2. **Compact operators.**

THEOREM 1 (ROSENTHAL [6]). *If* K *is a compact operator, then* K *has property* (P).

PROOF. Since K is compact, we may assume that the underlying Hilbert space H is separable.

Let $\mathcal{A}$ be a reductive algebra such that $K \in \mathcal{A}'$. Let $\mathcal{Z}$ be an abelian von Neumann algebra which is maximal with respect to the property $\mathcal{A} \subseteq \mathcal{Z}'$. Let $\mathcal{E}$ be the set of all (hermitian) projections in $\mathcal{Z}$. We call $E \in \mathcal{Z}$ an *atom* if $E \neq 0$ and if $0 \leqslant F \leqslant E$ with $F \in \mathcal{E}$ implies either $F = 0$ or $F = E$. Let $\{E_n\}$ be the set of all atoms of $\mathcal{E}$. Note that $E_nE_m = 0$ if $n \neq m$. Let $F = I - \Sigma_n E_n$. For $n \neq m$, we have $E_nKE_m \in \mathcal{A}'$ and $(E_nKE_m)^2 = 0$. By Lemma 1, we have $E_mK^*E_n \in \mathcal{A}'$. Similarly $FK^*E_n \in \mathcal{A}'$ and $E_nK^*F \in \mathcal{A}'$. From $E_nKE_n \in \mathcal{A}'$ we have $E_nK|E_nH \in (\mathcal{A}|E_nH)'$. Since E_nK is compact, we must have either $E_nKE_n = 0$ or $\dim E_nH = 1$; for otherwise, by Lomono-

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sov [4], $\mathcal{A}|E_nH$ would be intransitive, contradicting the maximality of $\mathcal{Z}$ and the fact that E_n is an atom. Now it remains to show that $FK^*|FH \in (\mathcal{A}|FH)'$. For convenience, we may assume that $F = I$. Thus, $\mathcal{E}$ has no atom. In other words, given $\varepsilon > 0$, $x \in H$ and $0 \neq F \in \mathcal{E}$, there exists $F' \in \mathcal{E}$ such that $0 \neq F' \lneq F$ and $\|F'x\| < \varepsilon$.

Let x be a unit vector and $\varepsilon > 0$. Let

$\mathcal{F} = \{F \in \mathcal{E}$: either $F = 0$ or there is a finite partition $0 = F_0 < F_1 < \cdots < F_m = F$ with $F_k \in \mathcal{E}$ such that $\|(F_k - F_{k-1})x\| < \varepsilon$ for $k = 1, \ldots, m\}$.

By Zorn's lemma, $\mathcal{F}$ has a maximal element. Since $\mathcal{E}$ has no atom, the maximal element must be I.

Next, let $\{x_n\}$ be a dense sequence of H. We can construct a sequence of partitions $\mathcal{P}_n = \{0 = F_0^{(n)} < \cdots < F_{m_n}^{(n)} = I\}$ in $\mathcal{E}$ such that

(1) $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \ldots$.

(2) $\|(F_k^{(n)} - F_{k-1}^{(n)})x_n\| < 1/n$ for $k = 1, \ldots, m_n$.

Write $E_k^{(n)} = F_k^{(n)} - F_{k-1}^{(n)}$. Since $\{x_n\}$ is dense in H, for each x in H, we have $\max_{1 \leqslant k \leqslant m_n} \|E_k^{(n)}x\| \to 0$ as $n \to \infty$. We assert that

$$\left\| \sum_{k=1}^{m_n} F_k^{(n)} K F_k^{(n)} \right\| = \max_k \|F_k^{(n)} K F_k^{(n)}\| \to 0$$

as $n \to \infty$. Since K is compact, it can be uniformly approximated by finite rank operators. Hence it suffices to consider the case when K is of the form $x \mapsto (x, a)b$. Now the assertion is clear.

Again, for $j \neq k$, $E_j^{(n)} K E_k^{(n)}$ is a nilpotent operator in $\mathcal{A}'$, so by Lemma 1, we have $E_k^{(n)} K^* E_j^{(n)} \in \mathcal{A}'$. Therefore

$$K^* = \lim_{k \to \infty} \sum_{k \neq j} E_k^{(n)} K^* E_j^{(n)} \in \mathcal{A}'. \quad \square$$

THEOREM 2. *If T is polynomially compact, then T has property* (P).

PROOF. Suppose that $p(T) = K$ is compact and $T \in \mathcal{A}'$ where p is a nonconstant polynomial. Since $K \in \mathcal{A}'$, we have $K^* \in \mathcal{A}'$ by the above theorem. Now K^*K is a compact hermitian operator; suppose $K^*K = \sum \lambda_k E_k$, where the $\{E_k\}$ are mutually orthogonal finite rank projections and the $\{\lambda_k\}$ are positive numbers. Let $\mathfrak{M}_k$ denote the range of E_k, $\mathfrak{M} = \ker p(T) = \ker K^*K$ and E be the projection of H onto $\mathfrak{M}$. Then $H = \mathfrak{M} \oplus \sum^{\oplus} \mathfrak{M}_k$. Obviously the E_k and E are in $\mathcal{A}'$. Again, by Lemma 1, it suffices to show that $E_k T^* E_k$ and ET^*E are in $\mathcal{A}'$. Since $ETE = TE$, we have $p(ETE) = p(T)E = KE = 0$ and hence ETE is algebraic. Also $ETE \in \mathcal{A}'$. By the corollary to Lemma 1, $ET^*E \in \mathcal{A}'$. Since $\dim \mathfrak{M}_k < \infty$, $E_k T E_k$ is algebraic, and hence by the same reasoning $E_k T^* E_k \in \mathcal{A}'$. This completes the proof. $\square$

3. **Operators quasi-similar to normals.** According to B. Sz.-Nagy and C. Foiaş [7], a linear, one-one and bounded mapping from a Hilbert space H_1 onto a dense subset of another Hilbert space H_2 is called a *quasi-affinity* from

H_1 into H_2; operators S_1 and S_2 defined on H_1 and H_2 respectively are called *quasi-similar* if there exist quasi-affinities $X: H_1 \to H_2$ and $Y: H_2 \to H_1$ such that $S_2X = XS_1$ and $YS_2 = S_1Y$.

THEOREM 3. *If T is quasi-similar to a normal operator, then T has property* (P).

PROOF. By assumption, there are quasi-affinities X and Y and a normal operator N such that $TX = XN$ and $YT = NY$. Let $\mathcal{A}$ be a reductive algebra with $T \in \mathcal{A}'$. We show that $T^* \in \mathcal{A}'$.

Suppose $\mathfrak{M}$ is an arbitrary spectral subspace of N. Let $\mathfrak{N}$ be the closure of $\{AXx: A \in \mathcal{A} \text{ and } x \in \mathfrak{M}\}$. It is easy to check that N commutes with YAX for every A in $\mathcal{A}$. Hence $Y\mathfrak{N} \subseteq \mathfrak{M}$. On the other hand, $\mathfrak{N}$ is an invariant subspace of $\mathcal{A}$ and hence reducing for $\mathcal{A}$. In particular, $A^*Xx \in \mathfrak{N}$ for $x \in \mathfrak{M}$ and $A \in \mathcal{A}$. Therefore $YA^*X\mathfrak{M} \subseteq Y\mathfrak{N} \subseteq \mathfrak{M}$. Hence YA^*X commutes with N for every A in $\mathcal{A}$. Now

$$YA^*TX = YA^*XN = NYA^*X = YTA^*X.$$

Since X and Y are quasi-affinities, we have $A^*T = TA^*$ for every A in $\mathcal{A}$. Hence $T^* \in \mathcal{A}'$. □

REMARK. From the proof we see that it suffices to assume that X is one-one and the range of Y is dense.

COROLLARY [2]. *If T is a reductive operator which is quasi-similar to a normal operator, then T is normal.*

PROOF. Let $\mathcal{A}$ be the weakly closed algebra generated by I and T. Then $\mathcal{A}$ is reductive and $T \in \mathcal{A}'$. By the above theorem, T^* commutes with T. Hence T is normal. □

Next we establish an algebraic version of Theorem 3.

THEOREM 4. *Suppose $\mathcal{A}$ is a von Neumann algebra on a Hilbert space H and $X: H \to K$ and $Y: K \to H$ are bounded linear maps. If $\mathcal{B} = \{A \in B(H): XAY \in \mathcal{A}$ is reductive$\}$, then $\mathcal{B}$ is selfadjoint.* (*Note that $\mathcal{B}$ is not necessarily an algebra.*)

PROOF. By the same argument as one we used in the proof of Theorem 3, we have $XB^*Y\mathfrak{M} \subseteq \mathfrak{M}$ for every reducing subspace $\mathfrak{M}$ and every operator B in $\mathcal{B}$. By the double commutant theorem, we have $XB^*Y \in \mathcal{A}$; that is, $B^* \in \mathcal{B}$ for $B \in \mathcal{B}$. □

COROLLARY. *If $\mathcal{A}$ is a von Neumann algebra on H_1, $\mathcal{B}$ is a reductive algebra on H_2 and there is an invertible bounded linear mapping $X: H_1 \to H_2$ such that $\mathcal{B} = X\mathcal{A}X^{-1}$, then $\mathcal{B}$ is also a von Neumann algebra and $\mathcal{A}$ and $\mathcal{B}$ are unitarily equivalent.*

PROOF. That $\mathcal{B}$ is a von Neumann algebra follows from Theorem 4 or the fact that similarity preserves reflexitivity of operator algebras. To show that $\mathcal{A}$ and $\mathcal{B}$ are unitarily equivalent, let $X = VP$ be the polar decomposition of X, where P is a positive and invertible operator on H_1 and V is an isometry from H_1 onto H_2. Then we have $\mathcal{B} = V(P\mathcal{A}P^{-1})V^{-1}$ and hence $P\mathcal{A}P^{-1}$ is selfadjoint. Therefore $P\mathcal{A}P^{-1} = P^{-1}\mathcal{A}P$ or $P^2\mathcal{A}P^{-2} = \mathcal{A}$. By Gardner's Invariance Theorem [3], we have $P\mathcal{A}P^{-1} = \mathcal{A}$. Hence $\mathcal{B} = V\mathcal{A}V^{-1}$. $\square$

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