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SPHERES IN E^3 WHICH ARE HOMOGENEOUS OVER A 0-DIMENSIONAL SET

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ABSTRACT. A 2-sphere S in Euclidean 3-space E^3 is defined to be homogeneous over the subset X of S if for each $p, q \in X$ there is a homeomorphism $h: E^3 \to E^3$ such that $h(S) = S$ and $h(p) = q$. It is shown that a 2-sphere S in E^3 is tame from one side provided S is locally tame modulo a tame 0-dimensional set C such that S is homogeneous over C. An example is described to show that it is necessary to require that C be tame.

It is not known whether a 2-sphere S is tame in E^3 if S is homogeneous in E^3; i.e., for each $p, q \in S$ there is a homeomorphism $h: E^3 \to E^3$ such that $h(S) = S$ and $h(p) = q$ [4, Question 4.5.3], [5, p. 312]. In this paper we show that a 2-sphere S in E^3 is tame from one side provided S is locally tame modulo a tame 0-dimensional set C such that S is homogeneous over C. (S is defined to be *homogeneous over the subset* X of S if for each $p, q \in X$ there is a homeomorphism $h: E^3 \to E^3$ such that $h(S) = S$ and $h(p) = q$.) Our proof of this theorem mainly involves showing that S can be locally spanned from one side [3, Theorem 7]. This theorem could be considered an extension of the theorem by Harrold and Moise [6] that a 2-sphere S in E^3 is tame from one side if S is locally tame modulo one point. We indicate that the Alexander horned sphere [1] is homogeneous over the tame Cantor set where the sphere is wild. We then notice, with an example described by Bing [2], that it is necessary to require, in the hypothesis of Theorem 1, that the 0-dimensional set C be tame in E^3.

We define a 2-sphere S in E^3 to be *tame from one side* if either $S \cup \operatorname{Int} S$ is a 3-cell or $S \cup \operatorname{Ext} S$ is homeomorphic to the closure of the complement of a round ball in E^3. A subset X of a 2-sphere S in E^3 is defined to be *tame* if X is a subset of some tame sphere in E^3. If S is a 2-sphere in E^3, U is a component of $E^3 - S$, and $x \in S$, we say that S can be *locally spanned at x from U* provided that for each $\varepsilon > 0$ there exist disks D and D' such that:

(1) $x \in \operatorname{Int} D \subset S$;
(2) $\operatorname{Bd} D = \operatorname{Bd} D'$;
(3) $\operatorname{Int} D' \subset U$;
(4) $\operatorname{Diam}(D \cup D') < \varepsilon$.

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We rely on [5] for various other definitions and notation.

THEOREM 1. *If S is a 2-sphere in E^3 and C is a 0-dimensional tame subset of S such that S is homogeneous over C and S is locally tame at each point of $S - C$, then S is tame from one side.*

PROOF. We will show that there is a component U of $E^3 - S$ such that S can be locally spanned from U at each point of S, and this will imply that S is tame from U [3, Theorem 7]. We consider S to be locally wild at each point of C, for otherwise the hypothesis would imply that S is locally tame. Since S is locally tame at each point of $S - C$, we may assume that $S - C$ is locally polyhedral [5, Theorem 4.4.5]. Let ε be a positive number and D_0 a disk in S such that $C \cap \operatorname{Int} D_0 \neq 0$ and $\operatorname{Diam} D_0 < \varepsilon/2$. Since C is tame, there exists a polyhedral sphere S_1 which has diameter less than $\varepsilon/2$ and encloses a point of $C \cap \operatorname{Int} D_0$ such that $S \cap S_1 \subset \operatorname{Int} D_0$ and $S_1 \cap C = 0$. We consider S and S_1 to be in relative general position so that each component of $S \cap S_1$ is a simple closed curve. By standard cut-and-paste techniques [5, p. 262], we adjust S_1 by removing any component of $S \cap S_1$ which is the boundary of a disk in $\operatorname{Int} D_0 - C$. With such an adjusted S_1, we obtain disks D_1 and E_1 such that:

(1) $E_1 \cap S = \operatorname{Bd} D_1 = \operatorname{Bd} E_1$;
(2) $D_1 \subset \operatorname{Int} D_0$ and $E_1 \subset S_1$;
(3) $\operatorname{Diam}(D_1 \cup E_1) < \varepsilon$;
(4) $C \cap \operatorname{Int} D_1 \neq 0$.

An iteration of this procedure enables us to obtain a null sequence $\{S_i\}$ of 2-spheres and null sequences $\{D_i\}$ and $\{E_i\}$ of disks such that for each i, requirements (1)–(4) are satisfied with S_1, D_1, and E_1 replaced with S_i, D_i, and E_i, respectively, and with D_0 replaced with D_{i-1}. Let $p = \cap_{i=1}^{\infty} D_i$. There is a component U of $E^3 - S$ such that infinitely many of the $\operatorname{Int} E_i$'s are in U. Thus S can be locally spanned at p from U. The hypothesis that S is homogeneous over C implies that S can be locally spanned from U at each point of C. With S locally tame at each point of $S - C$, we conclude that S is tame from U [3, Theorem 7].

THEOREM 2. *The Alexander horned sphere S is homogeneous over the Cantor set C of points where S is wild.*

INDICATION OF PROOF. In Figure 1, we indicate an Alexander horned sphere S which is tame from Ext S. We follow Bing's description in [2] where the D's are disks, the T's are tubes, and the K's are solid cylinders. A point x is in the Cantor set C of points where S is wild if and only if $x = K_{n_1} \cap K_{n_1 n_2} \cap K_{n_1 n_2 n_3} \cap \ldots$ and, for each i, n_i is either 1 or 2. Let p and q be two points of C. It is our purpose to show that there is a homeomorphism h: $E^3 \to E^3$ such that $h(S) = S$ and $h(p) = q$. We do this by identifying a sequence $\{h_i\}$ of homeomorphisms of E^3 onto itself which converges to the desired homeomorphism h. For convenience, we assume that $q = K_2 \cap K_{22} \cap K_{222} \cap \ldots$. If

$p \in K_2$, let h_1 be the identity homeomorphism on E^3. If $p \in K_1$, let h_1 be a homeomorphism of E^3 onto itself which revolves E^3 180° about an appropriate horizontal line such that $h_1(S) = S$ and $h_1(K_1) = K_2$. If $h_1(p) \in K_{22}$, let h_2 be the identity homeomorphism on E^3. If $h_1(p) \in K_{21}$, let h_2 be a homeomorphism of E^3 onto itself which is the identity except near K and which rotates both D_1 and D_2 180° in the same direction such that $h_2(T_{21}) = T_{22}$, $h_2(K_{21}) = K_{22}$, and $h_2(S) = S$. Similarly, h_3 is either the identity homeomorphism on E^3 or a homeomorphism h_3: $E^3 \to E^3$ which is the identity except near K_2 such that $h_3(K_{221}) = K_{222}$ and $h_3(S) = S$. This process can be continued to obtain a sequence $\{h_i\}$ of homeomorphisms which converges to the desired homeomorphism h.

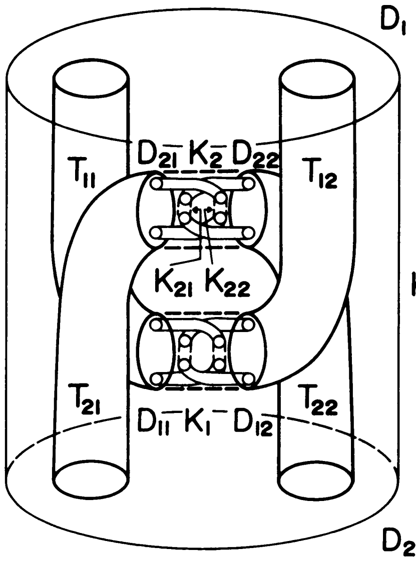

FIGURE 1

REMARK 1. It is necessary to require in Theorem 1 that the 0-dimensional set C be tame in E^3. The proof indicated above for Theorem 2 shows that for each p, $q \in C$, there is a homeomorphism h of $S \cup \operatorname{Int} S$ onto itself such that $h(p) = q$. Furthermore, the homeomorphism h could be constructed such that, for some $t \in \operatorname{Int} S$, $h(t) = t$. Bing [2] showed that S^3 results from sewing $S \cup \operatorname{Int} S$ to itself with the identity on S. Thus, by omitting the point t in one copy of $S \cup \operatorname{Int} S$, we have a 2-sphere S in E^3 such that (1) S is locally tame modulo a wild Cantor set C, (2) S is wild from each side at each point of C, and (3) S is homogeneous over C.

REMARK 2. We can adapt Theorem 1 to S^3 by requiring that the homeomorphism h, identified in the definition that S is homogeneous over the Cantor set C, should not interchange the components of $S^3 - S$. Without this additional requirement, we could change the construction indicated in Figure 1 such that the Cantor set C, of points where S is wild, is the union of two disjoint tame Cantor sets C_1 and C_2 with S locally tame from Int S at each point of C_1 and locally tame from Ext S at each point of C_2. This construction can be done so that S is homogeneous over C.

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