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SHORTER NOTES

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A SIMPLE PROOF OF RAMANUJAN'S ${}_1\psi_1$ SUM

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ABSTRACT. We give a simple proof of the ${}_1\psi_1$ sum using basic hypergeometric functions.

The purpose of this note is to show that Ramanujan's summation

$$(1)\qquad {}_1\psi_1\begin{pmatrix} a;q,x \\ b \end{pmatrix} =: \sum_{-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ba^{-1};q)_\infty (q;q)_\infty (qa^{-1}x^{-1};q)_\infty (ax;q)_\infty}{(b;q)_\infty (ba^{-1}x^{-1};q)_\infty (qa^{-1};q)_\infty (x;q)_\infty}$$

(where $|q| < 1$,

$$(a;q)_j = \prod_{n=0}^{\infty} \left(\frac{1 - aq^n}{1 - aq^{n+j}} \right) \quad \text{and} \quad (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) \Bigg)$$

follows immediately from the q-binomial theorem:

$$(2)\qquad \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty} \qquad [7, \text{p. } 92].$$

First, since

$$(3)\qquad {}_1\psi_1\begin{pmatrix} a;q,x \\ b \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n + \sum_{n=1}^{\infty} \frac{(qb^{-1};q)_n}{(qa^{-1};q)_n} \left(\frac{b}{ax} \right)^n,$$

we see that the ${}_1\psi_1$ is an analytic function of b provided $|q| < 1$, $|x| < 1$ and $|b| < |ax|$. To conclude we observe that (1) reduces to (2) whenever $b = q^m$, m a positive integer:

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$$
\begin{aligned}
{}_1\psi_1\left(\begin{matrix} a;q,x \\ q^m \end{matrix}\right) &= \sum_{k=1-m}^{\infty} \frac{(a;q)_k}{(q^m;q)_k} x^k = x^{1-m} \frac{(a;q)_{1-m}}{(q^m;q)_{1-m}} \sum_{k=0}^{\infty} \frac{(aq^{1-m};q)_k}{(q;q)_k} x^k \\
&= x^{1-m} \frac{(a;q)_{1-m}}{(q^m;q)_{1-m}} \cdot \frac{(axq^{1-m};q)_\infty}{(x;q)_\infty} \\
&= \frac{(q^m a^{-1};q)_\infty (q;q)_\infty (qa^{-1}x^{-1};q)_\infty (ax;q)_\infty}{(q^m;q)_\infty (q^m a^{-1}x^{-1};q)_\infty (qa^{-1};q)_\infty (x;q)_\infty}.
\end{aligned}
$$

Hence (1) is valid in general since it holds on a convergent sequence within the domain of analyticity.

The known proofs of (1) use either tricky rearrangements of series or functional equations, see Andrews [1], [2], Andrews and Askey [4], Hahn [5] and M. Jackson [6].

Finally we note that the ${}_1\psi_1$ sum includes the Jacobi triple product identity, see Andrews [3, pp. 169 – 172], as a limiting case, because

$$
\sum_{-\infty}^{\infty} q^{n^2} z^n = \lim_{c\to 0} {}_1\psi_1\left(\begin{matrix} -1/c;q^2,qxc \\ 0 \end{matrix}\right) = (q^2;q^2)_\infty(-qx^{-1};q^2)_\infty(-qx;q^2)_\infty .
$$

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