PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 1, March 1977

THE DERIVATIVE OF THE ATOMIC FUNCTION IS NOT IN $B^{2/3}$

CHARLES L. BELNA AND BENJAMIN MUCKENHOUPT[1]

ABSTRACT. H. A. Allen and C. L. Belna have shown that the derivative of the atomic function $A(z) = \exp[(z+1)/(z-1)]$ is in B^p for $0 < p < 2/3$, where B^p is the containing Banach space for the Hardy class H^p $(0 < p < 1)$. Here we show that $A'(z)$ does not belong to any of the other B^p spaces.

For $0 < p < 1$, the space B^p is the class of all functions $f(z)$ analytic in the open unit disk $\{z: |z| < 1\}$ for which

$$\|f\|_p \equiv \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|(1-r^2)^{1/p-2} d\theta dr < \infty.$$

H. A. Allen and C. L. Belna [2] proved that the derivative of the atomic function

$$A(z) \equiv \exp\left(\frac{z+1}{z-1}\right)$$

is in B^p for $0 < p < 2/3$. Combining the result of D. J. Newman and H. S. Shapiro [4, §2, p. 253] with that of P. L. Duren, B. W. Romberg, and A. L. Shields [3, Theorem 4, p. 41], we can easily show that $A'(z) \notin B^p$ for $4/5 \leqslant p < 1$. Here we fill the gap by showing that $A'(z) \notin B^{2/3}$.

We begin with the observation that

$$|A'(re^{i\theta})| = \frac{2}{1+r^2-2r\cos\theta}\exp\left(\frac{r^2-1}{1+r^2-2r\cos\theta}\right). \tag{1}$$

By restricting r and θ so that $\cos\theta < r < 1$ and $0 < \theta < \pi/2$, we have

$$\frac{1-r^2}{1+r^2-2r\cos\theta} < \frac{1-r^2}{1+r^2-2r^2} = 1 \tag{2}$$

and

$$\begin{aligned} \frac{2}{1+r^2-2r\cos\theta} &= \frac{2}{(1-r)^2+2r(1-\cos\theta)} \\ &> \frac{2}{(1-r)(1-\cos\theta)+2r(1-\cos\theta)} \\ &> \frac{1}{1-\cos\theta}. \end{aligned} \tag{3}$$

Received by the editors February 7, 1973 and, in revised form, March 9, 1976.
AMS (MOS) subject classifications (1970). Primary 30A78.

[1]Supported in part by NSF grant MPS 75-07596.

Using (2) and (3) in (1) shows that for $\cos\theta < r < 1$ and $0 < \theta < \pi/2$

$$|A'(re^{i\theta})| > \frac{1}{e(1-\cos\theta)}. \tag{4}$$

Finally, from (4) it follows that

$$e\cdot\|A'\|_{2/3} > e\cdot\int_0^{\pi/2}\int_{\cos\theta}^{1}|A'(re^{i\theta})|(1-r^2)^{-1/2}dr\,d\theta$$

$$> \int_0^{\pi/2}(1-\cos\theta)^{-1}\int_{\cos\theta}^{1}(1-r^2)^{-1/2}r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\sin\theta(1-\cos\theta)^{-1}d\theta = \infty.$$

Added July 1976. We note that during the delay in the publication of this paper, P. R. Ahern and D. N. Clark [1] improved on our result by showing that the derivative of any inner function having a nontrivial singular factor is not in $B^{2/3}$.

REFERENCES

1. P. R. Ahern and D. N. Clark, *On inner functions with B^p derivative*, Michigan Math. J. **23** (1976), 107–118.

2. H. A. Allen and C. L. Belna, *Singular inner functions with derivative in B^p*, Michigan Math. J. **19** (1972), 185–188. MR **45** #8844.

3. P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math. **238** (1969), 32–60. MR **41** #4217.

4. D. J. Newman and H. S. Shapiro, *The Taylor coefficients of inner functions*, Michigan Math. J. **9** (1962), 249–255. MR **26** #6371.

DEPARTMENT OF MATHEMATICS, WRIGHT STATE UNIVERSITY, CELINA, OHIO 45822

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903
(Current address of Benjamin Muckenhoupt)

Current address (C. L. Belna): Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802