THE DERIVATIVE OF THE ATOMIC FUNCTION IS NOT IN $B^{2/3}$

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Abstract. H. A. Allen and C. L. Belna have shown that the derivative of the atomic function $A(z) = \exp \left( \frac{z}{z - 1} \right)$ is in $B^p$ for $0 < p < 2/3$, where $B^p$ is the containing Banach space for the Hardy class $H^p$ ($0 < p < 1$). Here we show that $A'(z)$ does not belong to any of the other $B^p$ spaces.

For $0 < p < 1$, the space $B^p$ is the class of all functions $f(z)$ analytic in the open unit disk $\{ z : |z| < 1 \}$ for which

$$
\| f \|_p = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})||(1 - r^2)^{1/p - 2}\, d\theta \, dr < \infty.
$$

H. A. Allen and C. L. Belna [2] proved that the derivative of the atomic function

$$
A(z) \equiv \exp \left( \frac{z + 1}{z - 1} \right)
$$

is in $B^p$ for $0 < p < 2/3$. Combining the result of D. J. Newman and H. S. Shapiro [4, §2, p. 253] with that of P. L. Duren, B. W. Romberg, and A. L. Shields [3, Theorem 4, p. 41], we can easily show that $A'(z) \in B^p$ for $4/5 < p < 1$. Here we fill the gap by showing that $A'(z) \not\in B^{2/3}$.

We begin with the observation that

$$
|A'(re^{i\theta})| = \frac{2}{1 + r^2 - 2r\cos \theta} \exp \left( \frac{r^2 - 1}{1 + r^2 - 2r\cos \theta} \right).
$$

By restricting $r$ and $\theta$ so that $\cos \theta < r < 1$ and $0 < \theta < \pi/2$, we have

$$
\frac{1 - r^2}{1 + r^2 - 2r\cos \theta} < \frac{1 - r^2}{1 + r^2 - 2r^2} = 1
$$

and

$$
\frac{2}{1 + r^2 - 2r\cos \theta} = \frac{2}{(1 - r)^2 + 2r(1 - \cos \theta)}
$$

or

$$
\frac{2}{(1 - r)(1 - \cos \theta) + 2r(1 - \cos \theta)} > \frac{1}{1 - \cos \theta}.
$$

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Using (2) and (3) in (1) shows that for $\cos \theta < r < 1$ and $0 < \theta < \pi/2$

$$(4) \quad |A'(re^{i\theta})| > \frac{1}{e(1 - \cos \theta)}. \tag{4}$$

Finally, from (4) it follows that

$$e \cdot \|A'\|_{2/3} > e \cdot \int_0^{\pi/2} \int_{\cos \theta}^1 |A'(re^{i\theta})|(1 - r^2)^{-1/2} \, dr \, d\theta$$

$$> \int_0^{\pi/2} (1 - \cos \theta)^{-1} \int_{\cos \theta}^1 (1 - r^2)^{-1/2} r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin \theta (1 - \cos \theta)^{-1} \, d\theta = \infty. \tag{5}$$

Added July 1976. We note that during the delay in the publication of this paper, P. R. Ahern and D. N. Clark [1] improved on our result by showing that the derivative of any inner function having a nontrivial singular factor is not in $B^{2/3}$.

REFERENCES


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