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THE UNION OF TWO HILBERT CUBES MEETING IN A HILBERT CUBE NEED NOT BE A HILBERT CUBE[1]

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ABSTRACT. An example is given to verify the assertion of the title.

For some time there seems to have been a great deal of interest in the following question, which we shall herein answer in the negative: *If* $X = Q_1 \cup Q_2$, *where* $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$, *then is* $X \cong Q$? Here Q denotes the Hilbert cube. The question appears to have first been raised in [1], and has appeared several times since, notably in [3] and [5]. Anderson [1] showed that the answer is yes if $Q_1 \cap Q_2$ is a Z-set [2] in *each* of Q_1 and Q_2 and, more recently, Handel [10] has shown that the answer is yes if $Q_1 \cap Q_2$ is a Z-set in *either* Q_1 or Q_2.

If should be remarked that our example involves nothing more than an observation regarding already known facts. The first part of the observation is that the construction of Eaton's "dogbone" decomposition [9] of E^n can be carried out in Q, and that the resulting decomposition space X is not homeomorphic to Q. (This has evidently been noted by many others; see, e.g., the remark on p. 153 of [5].) The second part of the observation is that X can be decomposed as $Q_1 \cup Q_2$ where $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$.

The fact cited as the second part of our observation is a consequence of the following technical

LEMMA. *Suppose M is a Q-manifold, C a 0-dimensional compactum, and* F: $C \times [0, 1] \to M$ *an embedding such that if* $0 < \delta < 1$, $F(C \times [\delta, 1])$ *is a Z-set in M. Let G denote the upper semicontinuous decomposition of M whose nondegenerate elements are the members of the set* $\{F(\{x\} \times [0, 1])|x \in C\}$. *Then* $M/G \cong M$.

PROOF. It suffices to show that G satisfies the following version of the Bing Shrinking Criterion (cf. [6, p. 359, III]): if U is an open set containing the union of the nondegenerate elements of G and $\varepsilon > 0$, then there exists a homeomorphism h of M onto itself such that $h(p) = p$ for all $p \in M - U$ and diam $h(g) < \varepsilon$ for all $g \in G$. To verify this, suppose such a U and ε are

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given. Choose δ such that $0 < \delta < 1$ and diam $F(\{x\} \times [0, 2\delta]) < \varepsilon$ for all $x \in C$. Let $\mathcal{V}$ be an open cover of $U' = U - F(C \times \{0\})$ such that (1) if $x \in C$, then there exists $V \in \mathcal{V}$ such that $F(\{x\} \times [\delta, 1]) \subset V$, and (2) if h' is a homeomorphism of U' onto itself which is $\mathrm{St}^5(\mathcal{V})$-close to the identity, then h' extends, via the identity on $M - U'$, to a homeomorphism of M onto itself. Let $A = C \times (0, 1]$. Define f': $A \to U'$ by $f'(x, t) = F(x, t)$ for all $(x, t) \in A$ and g': $A \to U'$ by

$$g'(x, t) = \begin{cases} F(x, t) & \text{if } x \in C \text{ and } 0 < t \leqslant \delta, \\ F\left(x, \dfrac{\delta t + \delta - 2\delta^2}{1 - \delta}\right) & \text{if } x \in C \text{ and } \delta \leqslant t \leqslant 1. \end{cases}$$

Then $f'(A)$ and $g'(A)$ are Z-sets in U' and $f' \simeq_p g'$ via a proper homotopy limited by $\mathcal{V}$. It follows from Theorem 6.1 of [4] that there exists an invertible ambient isotopy H: $U' \times [0, 1] \to U'$ such that $H_0 = \mathrm{id}$, $H_1 f' = g'$, and H is limited by $\mathrm{St}^5(\mathcal{V})$. Extending H_1 to M via the identity on $M - U'$, we obtain a homeomorphism h of M onto itself satisfying the requirements of the shrinking criterion.

THE EXAMPLE. In [9], Eaton constructs a "dogbone" decomposition of E^n, $n \geqslant 3$, using a "ramified" version of Blankinship's construction of a wild Cantor set in E^n [7]. Wong [11] modified Blankinship's construction to obtain a Cantor set $K \subset Q$ such that $\pi_1(Q - K)$ is nontrivial. Using Wong's description, it is easy to see how to ramify the construction of K as in [9] to obtain a Cantor set L in Q having the necessary complications for the construction of [9] to be carried out. (The formalities of the ramification process are detailed in [8, §4].) In essence this construction can be thought of as identifying a Cantor set of Cantor sets, each embedded in Q as K is embedded in Q.

Now, $Q - L$ contains a Z-set homeomorphic to Q and, as previously mentioned, gluing two copies of Q along such a set yields a copy of Q. Identifying the resulting space with Q, we see that we may write $Q = Q' \cup Q''$, where $Q''' = Q' \cap Q'' \cong Q$ is a Z-set in each of Q' and Q'', and where there exist homeomorphisms h': $Q \to Q'$ and h'': $Q \to Q''$ such that $h'(L) \cap Q''' = \varnothing = h''(L) \cap Q'''$. We may further assume that the manifolds of the "special defining sequences" [8] used in defining $h'(L)$ and $h''(L)$ fail to intersect Q'''.

The reader who is familiar with [9] will now have no difficulty in seeing how to obtain an embedding H: $C \times [0, 1] \to Q$, where C is a Cantor set, such that

(1) for each $x \in C$, $H(\{x\} \times [0, 1])$ is a Z-set in Q,
(2) $H(C \times \{0\}) = h'(L)$ and $H(C \times \{1\}) = h''(L)$,
(3) $Q''' \cap H(C \times [0, 1]) = H(C \times \{\frac{1}{2}\})$,
(4) if $0 < \delta < \frac{1}{2}$, then $H(C \times [\delta, \frac{1}{2}])$ is a Z-set in Q' and

$$H\left(C \times \left[\tfrac{1}{2}, 1 - \delta\right]\right)$$

is a Z-set in Q'', and

(5) if G is the upper semicontinuous decomposition of Q whose nondegenerate elements are the members of the set $\{H(\{x\} \times [0, 1])|x \in C\}$, then $Q/G \not\cong Q$.

Let $X = Q/G$ and let Π: $Q \twoheadrightarrow X$ denote the natural projection. Letting $Q_1 = \Pi(Q')$ and $Q_2 = \Pi(Q'')$, it follows from the Lemma and condition (4) above that $Q_1 \cong Q \cong Q_2$. But $X = Q_1 \cup Q_2$ and, by condition (3), $Q_1 \cap Q_2 = \Pi(Q''') \cong Q$, so X provides the promised example.

We note that the above construction can be carried out in such a way that $H(C \times \{\frac{1}{2}\}) = C'$ is a Z-set in Q'''. Then there is a $(f\text{-}d)$ cap set in $Q''' - C'$, and it follows easily that there is a $(f\text{-}d)$ cap set in $Q_1 \cap Q_2$ which is a σ-Z-set in each of Q_1 and Q_2. On the other hand, we could arrange things so that $H(C \times \{\frac{1}{2}\}) = C''$ is embedded in Q''' as L is embedded in Q, with $H(C \times [0, \frac{1}{2}])$ providing a "mismatch" between $h'(L)$ and C''. In this case, there will be a disk D in $Q_1 \cap Q_2$ such that any disk "close" to D in $Q_1 \cap Q_2$ contains a Cantor set whose complement in Q_1 fails to be simply connected (the argument for this is contained in [8]). It follows that in this case there does not exist a $(f\text{-}d)$ cap set in $Q_1 \cap Q_2$ which is a σ-Z-set in Q_1.

ADDED IN PROOF. J. Quinn and R. Y. T. Wong have recently shown that the union of two Keller cubes (compact convex infinite dimensional subsets of l_2) meeting in a Keller cube is homeomorphic to Q.

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