ON THE HEIGHTS OF GROUP CHARACTERS

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Abstract. For a finite $p$-soluble group $G$ we derive a bound on the heights of the irreducible complex characters of $G$ lying in a $p$-block $B$. This bound depends on the prime $p$ and the exponent $d$ of a defect group of $B$. We show by examples that this bound is of the right order of magnitude.

Let $G$ be a finite group of order $p^e g_0$, where $p$ is a fixed prime, $e$ is an integer $> 0$, and $(g_0, p) = 1$. In the theory of modular representations, the characters of the irreducible complex representations of $G$ may be partitioned into disjoint sets, the so-called blocks of $G$ for the prime $p$. Associated with each block $B$ is a $p$-subgroup $D$ of $G$ of order $p^d$, unique up to conjugacy in $G$. $d$ is called the defect of the block $B$. If $\chi$ is an irreducible complex character of $G$, or as we shall say, an ordinary character of $G$, and $\chi$ lies in a block $B$, written $\chi \in B$, then $\chi$ has degree divisible by $p$ to the exponent $(e - d + h(\chi))$. The nonnegative integer $h(\chi)$ is called the height of $\chi$.

In [4] Fong proves the following: Let $G$ be a finite $p$-soluble group and $B$ be a block of $G$ for the prime $p$. Suppose that $B$ has defect group $D$ and let $Z(D)$ denote the centre of $D$. Then for each ordinary character $\chi \in B$ we have $h(\chi) \leq \nu_p([D : Z(D)])$ where $\nu_p(t)$ denotes the exponent of $p$ dividing $t$.

A slight modification of Fong's proof yields that for $d > 2$, $h(\chi)$ never exceeds $(d - 2)$. Brauer and Feit [2] obtain this bound for an arbitrary finite group. In this paper we prove the following result.

Theorem. Let $G$ be a finite $p$-soluble group with a block $B$ of defect $d > 2$. Then there exists a function $f(p, d)$ such that $h(\chi) \leq f(p, d)$ for all ordinary characters $\chi \in B$;

$$f(p, d) = \begin{cases} \left( \frac{3d - 4}{4} \right) & \text{if } p = 2, \\
\left( \frac{p^2 + 1}{p^2 - p + 1} \right) \left( \frac{d - 1}{2} \right) & \text{if } p \text{ is an odd Fermat prime,} \\
\left( \frac{p + 1}{p} \right) \left( \frac{d - 1}{2} \right) & \text{if } p \text{ is any other prime.} \end{cases}$$

We give examples to show that this bound is of the right order of magnitude.

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Lemma 1. Let $G$ be a finite $p$-soluble group and $B$ be a block of $G$ with defect group $D$, of order $p^d$. Suppose that $H < G$, then there exists an irreducible constituent of $\chi|_H$, say $\theta$, such $h(\chi) < h(\theta) + r_p(\mid G : H\mid)$. $h(\theta)$ denotes the height of $\theta$ over the block $b$ of $H$ with $\theta \in b$ and $h(\chi)$ denotes the height of $\chi$ over $B$.

Proof. Choose a series $G = G_1 \supset G_2 \supset \cdots \supset G_r = H$ such that $G_{i+1}$ is a maximal normal subgroup of $G_i$ for $i = 1, \ldots, r - 1$. Define $e_i = r_p(\mid G_i\mid)$. Now choose ordinary characters $\chi_1, \ldots, \chi_r$ such that $\chi_i$ is an ordinary character of $G_i$ and $\chi = \chi_1$, and $\chi_{i+1}$ is an irreducible constituent of $\chi_i$ restricted to $G_{i+1}$. Now there exist blocks $B_1, \ldots, B_r$ with $B_i$ a block of $G_i$ containing $\chi_i$ for $i = 1, \ldots, r$. Let $\theta = \chi_r$ and hence $B_r = b$. These conditions mean that $B_i$ covers $B_{i+1}$ in the sense of Brauer [1]. Finally let $D_i$ be a defect group of $B_i$ for each $i$ and suppose that $d_i = \text{defect of } B_i$. We have

$$r_p(\deg \chi_i) = e_i - d_i + h(\chi_i) \quad \text{for } i = 1, \ldots, r.$$ 

When $\mid G_i : G_{i+1}\mid$ is coprime to $p$ then by Clifford's theorem $r_p(\deg \chi_i)$ equals $r_p(\deg \chi_{i+1})$. Clearly $e_i = e_{i+1}$ and by [1, 2E] $d_i = d_{i+1}$. Thus $h(\chi_i) = h(\chi_{i+1})$. Otherwise $\mid G_i : G_{i+1}\mid = p$ and Clifford's theorem yields that

$$r_p(\deg \chi_i) < r_p(\deg \chi_{i+1}) + 1.$$ 

Also $e_i = e_{i+1} + 1$ and by [3], $d_i < d_{i+1} + 1$. We conclude that in this case $h(\chi_i) < h(\chi_{i+1}) + 1$.

Hence $h(\chi) = h(\chi_1) < h(\chi_r) + r_p(\mid G : H\mid) = h(\theta) + r_p(\mid G : H\mid)$ as required.

Lemma 2. If $G$ is a finite $p$-soluble group which is faithfully and irreducibly represented on a vector space $V$ of dimension $n$ over $GF(p)$ then $r_p(\mid G\mid)$ does not exceed $\lambda(p, n)$ where

$$\lambda(p, n) = \begin{cases} (n - 1) & \text{if } p = 2, \\ (np)/(p - 1) & \text{if } p \text{ is an odd Fermat prime,} \\ n/(p - 1) & \text{otherwise.} \end{cases}$$

Proof. $r_p(\mid G\mid) \leq n - 1$ by Huppert [7, Satz 14]. For odd $p$ a paper of Winter [8] yields that

$$r_p(\mid G\mid) \leq \sum_{i=0}^{\infty} \frac{n}{p^i(p - 1)} \quad \text{for } p \text{ Fermat,}$$

$$\sum_{i=1}^{\infty} \frac{n}{p^i} \quad \text{for } p \text{ not Fermat.}$$

Since for $|x| < 1$, $\sum_{i=0}^{\infty} x^i = 1/(1 - x)$ our lemma follows easily.

Lemma 3. If $G$ is a finite $p$-soluble group with $O_p(G) \cong O_p(G)\times O_p(G)$ and if $\mid O_p(G) : \Phi(O_p(G))\mid = p^m$ where $\mid O_p(G)\mid = p^n$ then

$$r_p(\mid G\mid) \leq m + \lambda(p, n).$$
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Proof. By [5, 1.2.5] \( G/O_{p^p}(G) \) is faithfully represented on \( O_{p^p}(G)/F \) where \( F/O_{p^p}(G) = \Phi(O_{p^p}(G)/O_{p^p}(G)) \). Thus under our hypotheses \( G/O_{p^p}(G) \) is faithfully represented on \( O_{p^p}(G)/\Phi(O_{p^p}(G)) \). Let \( L_1, \ldots, L_s \) denote the \( p \)-chief factors of \( G \) lying between \( O_{p^p}(G) \) and \( \Phi(O_{p^p}(G)) \). Then \( C = \bigcap_{i=1}^s C_i(L_i) \supset O_{p^p}(G) \) since \( O_{p^p}(G) \) centralizes all \( p \)-chief factors. For each \( i = 1, \ldots, s \) we have that \( G/C_i(L_i) \) is a faithful irreducible subgroup of \( GL(n_i, p) \) where \( n_i \) is just the rank of \( L_i \). Since \( G/O_{p^p}(G) \) is merely a group of automorphisms of a \( p \)-group which stabilizes a normal series for that group we conclude that \( C/O_{p^p}(G) \) is a \( p \)-group and thus \( C = O_{p^p}(G) \).

Now \( G/O_{p^p}(G) \) is isomorphic to a subgroup of \( G/C_G(L_1) \times \cdots \times G/C_G(L_s) \) so in particular

\[
\nu_p\left(|G : O_{p^p}(G)|\right) \leq \sum_{i=1}^s \nu_p\left(|G : C_G(L_i)|\right).
\]

Since \( \lambda(p, k) \) is linear in the second variable, and using Lemma 2 we see that \( \nu_p(|G|) \leq m + \lambda(p, n) \) as required.

Proof of Theorem. We proceed by induction on the order of \( G \). By [4, 2B and 2D] we may assume that all blocks of \( G \) have maximal defect, so \( d = \nu_p(|G|) \). Furthermore \( O_p(G) \) is cyclic and central in \( G \) so \( O_{p^p}(G) = O_p(G) \times O_p(G) \). Let \( H = O_p(G) \) and set \( |H| = p^m \) and \( |H : \Phi(H)| = p^n \). Let \( \theta \) be an irreducible constituent of \( \chi|_H \). Now \( \nu_p(|G : H|) = d - m \) and thus by Lemma 1 \( h(\chi) \leq h(\theta) + d - m \). We consider two possibilities:

(a) \( H \) is abelian. In this case \( h(\theta) = 0 \) and by Lemma 3, since \( n \leq m \) we have that \( d \leq m + \lambda(p, m) \). Then \( \lambda(p, 2, m) = m - 1 \) and so \( d \leq 2m - 1 \). Hence \( h(\chi) \leq (d - 1)/2 \). For \( p \) an odd Fermat prime a similar calculation yields \( h(\chi) \leq (pd)/(p^2 - p + 1) \). Finally for \( p \) odd and not Fermat, using Lemma 2 again we deduce that \( h(\chi) \leq d/p \). These three bounds are less than the ones appearing in the statement of the theorem.

(b) \( H \) is nonabelian. Now \( h(\theta) \leq (m - 1)/2 \) since \( \theta \) is a character of a nonabelian \( p \)-group of order \( p^m \). Also \( H \) nonabelian implies that \( n \leq m - 1 \). Thus \( h(\chi) \leq d - (m + 1)/2 \) by Lemma 1 and \( d \leq m + \lambda(p, m - 1) \) by Lemma 3. For \( p = 2, d < 2(m - 1) \) and so \( h(\chi) \leq (3d - 4)/4 \). Finally for non-Fermat primes \( p, d \leq (mp - 1)/(p - 1) \) and a brief calculation yields

\[
h(\chi) \leq ((p + 1)/p)((d - 1)/2).
\]

Our theorem is now proved.

The theorem is best possible in the following sense: Given an odd integer \( d \geq 1 \) choose \( p \) to be a prime with \( p > d \). Now there exists an extraspecial \( p \)-group of order \( p^d \); this group possesses an ordinary character of height \( 1/2(d - 1) \). We have so chosen things that \( 1/2(d - 1) \) is the greatest integer less than \( f(p, d) \). We give less trivial examples for \( p = 2 \) and \( p = 3 \) below.

(1) Let \( G_i \cong GL(2, 3) \), the group of \( 2 \times 2 \) matrices over the Galois field of three elements, for \( i = 1, 2, \ldots, n \). Form the central product \( G(n) = G_1 \times G_2 \times \cdots \times G_n \). Let \( Z(G_i) = gp(z_i, z_i^{-1}) = 1 \) then \( G(n) \cong (G_1 \times G_2 \times \cdots \times G_n)/\Delta \) where \( \Delta = gp(z_1z_2^{-1}, z_2z_3^{-1}, \ldots, z_nz_1^{-1}) \). Now \( G \)
is a 2-soluble group and \(O_2/(G(n)) = 1\). Thus \(G(n)\) possesses one block for the prime 2 and hence this has defect \(d = 3n + 1\). \(G_i\) has a character \(\theta_i\) of degree four such that \(\theta_i(z_i) = -4\). Form \(\theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n = \chi\), an irreducible character of \(G(n)\), since one easily checks that \(\Delta \leq \text{Ker} \chi\). We have that \(h(\chi) = 2n = 2(d - 1)/3\).

(2) Let \(E\) be the extraspecial 3-group of order 27 and exponent 3. Since \(\text{SL}(2,3)\) is isomorphic to a subgroup of the automorphism group of \(E\), we may form the semidirect product of \(E\) by \(\text{SL}(2,3)\). Denote this group by \(H\). \(H\) has a centre of order 3 and furthermore has an irreducible character of degree 9, say \(\alpha\). If \(Z(H) = \text{gp}\{z_i; z_i^3 = 1\}\) then \(\alpha(z_i) = 9\omega\) and \(\alpha(z_i^2) = 9\omega^2\) where \(\omega\) is a primitive cube root of unity. As in the previous example we construct \(H(n)\), the central product of \(n\) copies of \(H\). \(H(n)\) is a 3-soluble group in which \(O_3(H(n)) = 1\) and so \(H(n)\) has a unique block for the prime 3 and this has defect \(d = 3n + 1\). The character \(\psi = \text{tensor product of } n \text{ copies of } \alpha\), is an irreducible character of \(H(n)\). \(\psi\) has height \(2n = 2(d - 1)/3\).

We have shown that if \(g(p,d)\) is the precise bound on the heights of \(p\)-soluble groups then

\[
2(d - 1)/3 \leq g(2,d) \leq 3(d - 1)/4,
\]
\[
2(d - 1)/3 \leq g(3,d) \leq 5(d - 1)/7.
\]

In fact there exist examples for all primes \(p\) of \(p\)-soluble groups \(G\) with a \(p\)-block \(B\), of defect \(d\), containing an ordinary character of height exceeding \((d - 1)/2\).

REFERENCES