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OPERATORS COMMUTING WITH A COMPACT QUASIAFFINITY[1]

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ABSTRACT. We present an example of a nonbiquasitriangular operator which commutes with a compact quasiaffinity.

Let $\mathcal{H}$ denote a complex, separable Hilbert space, $B(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$. Recall that $X \in B(\mathcal{H})$ is called a quasiaffinity provided X and X^* have trivial null spaces; we will denote by K_a the set of compact quasiaffinities. In a recent paper [2], Foias, Pearcy, and Voiculescu consider the problem of which operators can commute with an operator $K \in K_a$. They show that if T is a nonbiquasitriangular operator such that $TK = KT$ and $K \in K_a$, then the index of $T - \lambda$ cannot be finite for any scalar λ such that $T - \lambda$ is a semi-Fredholm operator. To complete the discussion, they give an example (Proposition 5.3) of a semi-Fredholm operator T with infinite index and $K \in K_a$ such that T and K commute. However, this example is based upon the incorrect statement that the Volterra operator V is similar to $V/2$ (see [3]). In fact, it follows from a result of Kalisch [4, Lemma 5], that if $X \in B(\mathcal{H})$ is similar to $X/2$, then X must be nilpotent. Thus, there is no compact quasiaffinity which can replace the Volterra operator in their example.

In this note, we show that a slight modification of the construction presented in [2] can be used to provide the desired example. We define the semi-Fredholm operator T and the operator $K \in K_a$ on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ via the operator matrices

$$T = \begin{pmatrix} 0 & & & & 0 \\ X_1 & 0 & & & \\ & X_2 & 0 & & \\ & & X_3 & 0 & \\ 0 & & & \ddots & \ddots \end{pmatrix}, \qquad K = \begin{pmatrix} K_1 & & 0 & \\ & K_2 & & \\ 0 & & K_3 & \\ & & & \ddots \end{pmatrix}$$

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In order for T to be semi-Fredholm, it is sufficient that X_j be invertible for all j, and $\sup_j\{\|X_j\|, \|X_j^{-1}\|\} < \infty$. Furthermore, if $K_j \in K_a$ for all j, and $\|K_j\| \to 0$, then $K \in K_a$. Finally, a computation shows that T and K commute provided $K_{j+1} = X_j K_j X_j^{-1}$.

Let $\{e_m\}$, $m = 0, \pm 1, \pm 2, \ldots$, be an orthonormal basis for $\mathcal{H}$. We define K_1 to be a compact bilateral weighted shift on $\mathcal{H}$; i.e., $K_1 e_m = w_{1,m} e_{m+1}$, where $w_{1,m} > 0$ and $w_{1,m} \to 0$. Thus, $K_1 \in K_a$. Since K_1 is compact, $\lim_{m\to\pm\infty} w_{1,m} = 0$, and thus there exists an index m_0 such that $w_{1,m_0} = \max_m\{w_{1,m}\}$. We now define $K_2 \in K_a$ via $K_2 e_m = w_{2,m} e_{m+1}$, where $w_{2,m} = w_{1,m}$ for $m \neq m_0$, and $w_{2,m_0} = w_{1,m_0}/2$. Theorem 2 of [5] says that K_1 and K_2 are similar, and if $K_2 = X_1 K_1 X_1^{-1}$, then X_1 can be chosen such that $\max\{\|X_1\|, \|X_1^{-1}\|\} \leqslant 2$. If we now define K_{j+1} from K_j in the same fashion, then clearly $\|K_j\| \to 0$ and $\sup_j\{\|X_j\|, \|X_j^{-1}\|\} \leqslant 2$. Hence, Proposition 5.3 of [2] is established.

As a final comment, we point out that a result of R. G. Douglas [1, Theorem 8] shows that T cannot be an isometry.

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