LONGITUDE SURGERY ON GENUS 1 KNOTS

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Abstract. Let \( l(K) \) be the closed 3-manifold obtained by longitude surgery on the knot manifold \( K \). Let \( C \) be the cube with holes obtained by removing an open regular neighborhood of a minimal spanning surface in \( K \). The main result of this paper is that if \( K \) is of genus 1 and the longitude of \( K \) is in each term of the lower central series for \( \Pi_1(C) \), then \( l(K) \) is not homeomorphic to the connected sum of \( S^1 \times S^2 \) and a homotopy 3-sphere. In particular, this implies we cannot obtain the connected sum of \( S^1 \times S^2 \) and a homotopy 3-sphere by longitude surgery on any pretzel knot of genus 1.

Let \( K \) be a knot manifold, i.e., \( K \) is the complement of an open regular neighborhood of a piecewise linear simple closed curve in \( S^3 \). We assume that \( \Pi_1(K) \neq \mathbb{Z} \) (the knot is really knotted). The longitude \( l \) of \( K \) is the unique (up to isotopy in \( \partial K \)) simple closed curve in \( \partial K \) which is homologous to zero in \( K \) but bounds no disk in \( \partial K \). It has been conjectured that if we glue a solid torus \( T \) to \( K \) along the boundary of each in such a way that the meridian of \( T \) (a simple closed curve bounding a disk in \( T \) but none in \( \partial T \)) is identified with \( l \), we do not obtain \( S^1 \times S^2 \), or, more generally, we do not obtain a closed 3-manifold which is the connected sum of \( S^1 \times S^2 \) and a homotopy 3-sphere (denote such a manifold by \( (S^1 \times S^2)' \)). We refer to the above operation as longitude surgery on \( K \) and denote the resulting closed orientable manifold by \( l(K) \) (note that \( H_2(l(K)) = \mathbb{Z} \)). In this note we give an algebraic condition which implies that for genus 1 knots, 

\[ l(K) \neq (S^1 \times S^2)' \]

Louise Moser [6] has shown that \( l(K) \neq (S^1 \times S^2)' \) for \( K \) a doubled knot, a composite knot or a knot with nontrivial Alexander polynomial.

Let \( S \) be an orientable, incompressible [3] surface in \( K \) such that \( \partial S \) is \( K \)'s longitude. Since such an \( S \) always exists for any knot \( K \) (see [2]), we assume that the genus of \( S \) is smallest possible. Such an \( S \) will be referred to

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as a minimal spanning surface for $K$ and the genus of $K$ is the genus of $S$. Let $C$ be the cube with holes obtained by removing from $K$ an open regular neighborhood of $S$ in $K$. There is a natural decomposition of $\text{Bd} \; C$ into three submanifolds $S_1$, $S_2$, and $A_0$, where $S_1$, $S_2$ are copies of $S$, one on either side of $S$, and $A_0$ is the annulus which connects $S_1$ to $S_2$ and its center-line is a longitude of $K$. Let $G$ be a group, $G_i$ the $i$th term in the lower central series and $G_\omega = \bigcap_1^\infty G_i$ (see [4] or [5]).

We need two lemmas before proving Theorem 1.

**Lemma 1.** If $l(K) = (S_1 \times S_2)'$, then $K$ contains a properly embedded connected planar surface $X$ such that the number of components of $\text{Bd} \; X$ is odd, each component of $\text{Bd} \; X$ is parallel to $K$'s longitude $l$, and $X$ is incompressible in $K$.

**Proof.** Suppose $l(K) = (S_1 \times S_2)'$, that is, longitude surgery on $K$ produces a closed orientable 3-manifold which factors into $S_1 \times S_2$ and a homotopy 3-sphere. Regard the knot manifold $K$ and the attached solid torus $T$ as submanifolds of the closed, orientable manifold $l(K)$. Put $p \times S^2 (\subset S^1 \times S^2, p \in S^1)$ in general position relative to $K$. It follows that we may isotope $p \times S^2$ until $T \cap (p \times S^2)$ consists only of meridional disks of $T$ and, hence, $(p \times S^2) \cap \text{Bd} \; K$ consists of longitudes of $K$. Since $p \times S^2$ does not separate in $l(K)$ and $(p \times S^2) - T$ is connected, the number of components of $(p \times S^2) \cap \text{Bd} \; K$ is odd [1]. If $X' = (p \times S^2) \cap K$ is compressible in $K$, then there would exist a disk $\Delta$ in $K$ such that $\Delta \cap X' = \text{Bd} \; \Delta$ and $\text{Bd} \; \Delta$ separates some component of $\text{Bd} \; X'$ from another in $X'$. We may then remove a small open regular neighborhood of $\text{Bd} \; \Delta$ from $X'$ and fill in the two resulting holes by disjoint parallel copies of $\Delta$. We repeat this process a finite number of times, choosing at each stage the planar surface with an odd number of boundary components. When we cannot go any further, we have the desired planar surface $X$ of the hypothesis.

**Lemma 2.** Suppose $M$ is any 3-manifold contained in $S^3$ and $X$ is a connected planar surface properly embedded in $M$ such that the number of components of $\text{Bd} \; X$ is odd and each component of $\text{Bd} \; X$ lies in an annulus $Y$, $Y \subset \text{Bd} \; M$, and is parallel to $Y$'s center-line. Then the center-line of $Y$ is in $(\Omega_1(M))_\omega$.

**Proof.** Because the number of components of $\text{Bd} \; X$ is odd and because each of them is parallel to $Y$'s center-line $y$, it follows that we can add subannuli $Y'_1, \ldots, Y'_m$ of $Y$ to $X$ and adjust the result slightly to form an orientable surface $X'$ in $M$ with one boundary component $y$. Let $x_0$ be a point in $y$. Then, in $X'$, there is a collection of simple closed curves $\alpha_i, \beta_i$, $i = 1, \ldots, m$, having only the one point $x_0$ in common and such that each $\alpha_i$ is contained in $X$ and separates one boundary component of $Y'_i$ from the other in $X$ but fails to separate the boundary components of each $Y'_j, j \neq i$, in $X$, and each $\beta_i$ crosses the handle in $X'$ formed by $Y'_i$ once and otherwise $\beta_i \cap Y'_j = \emptyset$, $i \neq j$. Since $X$ is planar, it follows that $[y] = \emptyset$. 

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\[ \prod_{i=1}^{m} [c_i][y][c_i]^{-1}[\beta_i][c_i][y]^{-1}[c_i]^{-1}[\beta_i]^{-1}. \]

The product (1) implies \([y]\) is in the second term of the lower central series (the commutator subgroup) of \(\Pi_1(M, x_0)\). Since the terms in the lower central series are normal subgroups, \([c_i][y][c_i]^{-1}\) is also in the second term of the lower central series. Hence (1) implies \([y]\) is in the third term of the lower central series. Proceeding in this manner we see that \([y] \in (\Pi_1(M))_\omega\).

**Theorem 1.** If \(K\) is of genus 1 and \(l \not\in (\Pi_1(C))_\omega\), then \(l(K) \neq (S^1 \times S^2)'\).

**Proof.** Suppose \(l(K) = (S^1 \times S^2)'\). By Lemma 1, there exists a planar surface \(D\) properly embedded in \(K\) such that the number of components of \(\partial D\) is odd, each component of \(\partial D\) is parallel to \(K\)'s longitude \(l\) and \(D\) is incompressible in \(K\). Put \(D\) in general position relative to the minimal spanning surface \(S\) of genus 1. If any simple closed curve \(x\) of \(S \cap D\) bounds a disk in \(S\), then, since \(D\) is incompressible, \(x\) bounds a disk in \(D\). Suppose \(x\) is innermost relative to \(S\), i.e., the disk \(x\) bounds in \(S\) contains no points of \(D\) in its interior. Then we may replace the disk \(x\) bounds in \(D\) by the one it bounds in \(S\) and, pushing the new \(D\) off \(S\), we obtain a \(D \cap S\) with fewer components. Since the above statements are true for \(S\) interchanged with \(D\), we may suppose that no simple closed curve of \(S \cap D\) bounds a disk in either \(S\) or \(D\). If \(S \cap D = \emptyset\), then \(D \subset C\) and, by Lemma 2, \(l \in (\Pi_1(C))_\omega\), contradicting our hypothesis. If \(x\) is a simple closed curve of \(S \cap D\) which separates \(S\), then, since \(S\) is of genus 1, \(x\) and \(\partial S\) cobound an annulus \(Z\) in \(S\). We may suppose \(\text{Int} Z \cap D = \emptyset\). We then obtain two planar surfaces \(D', D''\) by adding two copies of \(Z\) to the two components of \(D - x\), respectively, and adjusting by pushing both off \(Z\). Now one of the resulting two surfaces \(D', D''\), say \(D'\), has an odd number of boundary components and \(D' \cap S\) has fewer components than \(D \cap S\). Suppose then that \(x\) does not separate \(S\) and that \(x\) is innermost on \(D\), i.e., one component \(D'\) of \(D - x\) contains no points of \(S\). Let \(N(x)\) be a small regular neighborhood of \(x\) in \(S\). Let \(S' = S - \text{Int} N(x)\) and form a new planar surface \(D_0\) by adding two parallel copies of \(D'\) to \(S'\), i.e., fill in the two holes of \(S'\) with copies of \(D'\) to form \(D_0\). Note that we may push \(D_0\) off \(S\) so that \(D_0 \cap S = \emptyset\) and that \(D_0\) has an odd number of boundary components. Now in all cases we have the contradiction that \(l \in (\Pi_1(C))_\omega\) and the proof is complete.

If \(F\) is a free group, then \(F_\omega = 1\) [5, pp. 311–312]; thus, we have the following corollary.
Corollary 1. If $K$ is of genus 1 and $C$ is a cube with two handles ($K$ has an algebraically unknotted, minimal spanning surface), then $l(K) \neq (S^1 \times S^2)'$.

Corollary 1 implies that for every pretzel knot $K$ of genus 1 [7] we have $l(K) \neq (S^1 \times S^2)'$ (Theorem 1 of [6] does not apply to all pretzel knots since some have trivial Alexander polynomial.)

References


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