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# LONGITUDE SURGERY ON GENUS 1 KNOTS

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ABSTRACT. Let $l(K)$ be the closed 3-manifold obtained by longitude surgery on the knot manifold $K$. Let $C$ be the cube with holes obtained by removing an open regular neighborhood of a minimal spanning surface in $K$. The main result of this paper is that if $K$ is of genus 1 and the longitude of $K$ is in each term of the lower central series for $\Pi_1(C)$, then $l(K)$ is not homeomorphic to the connected sum of $S^1 \times S^2$ and a homotopy 3-sphere. In particular, this implies we cannot obtain the connected sum of $S^1 \times S^2$ and a homotopy 3-sphere by longitude surgery on any pretzel knot of genus 1.

Let $K$ be a knot manifold, i.e., $K$ is the complement of an open regular neighborhood of a piecewise linear simple closed curve in $S^3$. We assume that $\Pi_1(K) \neq Z$ (the knot is really knotted). The longitude $l$ of $K$ is the unique (up to isotopy in Bd $K$) simple closed curve in Bd $K$ which is homologous to zero in $K$ but bounds no disk in Bd $K$. It has been conjectured that if we glue a solid torus $T$ to $K$ along the boundary of each in such a way that the meridian of $T$ (a simple closed curve bounding a disk in $T$ but none in Bd $T$) is identified with $l$, we do not obtain $S^1 \times S^2$, or, more generally, we do not obtain a closed 3-manifold which is the connected sum of $S^1 \times S^2$ and a homotopy 3-sphere (denote such a manifold by $(S^1 \times S^2)'$). We refer to the above operation as longitude surgery on $K$ and denote the resulting closed orientable manifold by $l(K)$ (note that $H_2(l(K)) = Z$). In this note we give an algebraic condition which implies that for genus 1 knots,

$$l(K) \neq (S^1 \times S^2)'.$$

Louise Moser [6] has shown that $l(K) \neq (S^1 \times S^2)'$ for $K$ a doubled knot, a composite knot or a knot with nontrivial Alexander polynomial.

Let $S$ be an orientable, incompressible [3] surface in $K$ such that Bd $S$ is $K$'s longitude. Since such an $S$ always exists for any knot $K$ (see [2]), we assume that the genus of $S$ is smallest possible. Such an $S$ will be referred to

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as a minimal spanning surface for $K$ and the genus of $K$ is the genus of $S$. Let $C$ be the cube with holes obtained by removing from $K$ an open regular neighborhood of $S$ in $K$. There is a natural decomposition of Bd $C$ into three submanifolds $S_1$, $S_2$, and $A_0$, where $S_1$, $S_2$ are copies of $S$, one on either side of $S$, and $A_0$ is the annulus which connects $S_1$ to $S_2$ and its center-line is a longitude of $K$. Let $G$ be a group, $G_i$ the $i$th term in the lower central series and $G_\omega = \bigcap_{i=1}^{\infty} G_i$ (see [4] or [5]).

We need two lemmas before proving Theorem 1.

LEMMA 1. *If* $l(K) = (S^1 \times S^2)'$, *then* $K$ *contains a properly embedded connected planar surface* $X$ *such that the number of components of* Bd $X$ *is odd, each component of* Bd $X$ *is parallel to* $K$*'s longitude* $l$, *and* $X$ *is incompressible in* $K$.

PROOF. Suppose $l(K) = (S^1 \times S^2)'$, that is, longitude surgery on $K$ produces a closed orientable 3-manifold which factors into $S^1 \times S^2$ and a homotopy 3-sphere. Regard the knot manifold $K$ and the attached solid torus $T$ as submanifolds of the closed, orientable manifold $l(K)$. Put $p \times S^2$ ($\subset S^1 \times S^2, p \in S^1$) in general position relative to $K$. It follows that we may isotope $p \times S^2$ until $T \cap (p \times S^2)$ consists only of meridianal disks of $T$ and, hence, $(p \times S^2) \cap$ Bd $K$ consists of longitudes of $K$. Since $p \times S^2$ does not separate in $l(K)$ and $(p \times S^2) - T$ is connected, the number of components of $(p \times S^2) \cap$ Bd $K$ is odd [1]. If $X' = (p \times S^2) \cap K$ is compressible in $K$, then there would exist a disk $\Delta$ in $K$ such that $\Delta \cap X' =$ Bd $\Delta$ and Bd $\Delta$ separates some component of Bd $X'$ from another in $X'$. We may then remove a small open regular neighborhood of Bd $\Delta$ from $X'$ and fill in the two resulting holes by disjoint parallel copies of $\Delta$. We repeat this process a finite number of times, choosing at each stage the planar surface with an odd number of boundary components. When we cannot go any further, we have the desired planar surface $X$ of the hypothesis.

LEMMA 2. *Suppose* $M$ *is any* 3*-manifold contained in* $S^3$ *and* $X$ *is a connected planar surface properly embedded in* $M$ *such that the number of components of* Bd $X$ *is odd and each component of* Bd $X$ *lies in an annulus* $Y$, $Y \subset$ Bd $M$, *and is parallel to* $Y$*'s center-line. Then the center-line of* $Y$ *is in* $(\Pi_1(M))_\omega$.

PROOF. Because the number of components of Bd $X$ is odd and because each of them is parallel to $Y$'s center-line $y$, it follows that we can add subannuli $Y_1', \ldots, Y_m'$ of $Y$ to $X$ and adjust the result slightly to form an orientable surface $X'$ in $M$ with one boundary component $y$. Let $x_0$ be a point in $y$. Then, in $X'$, there is a collection of simple closed curves $\alpha_i$, $\beta_i$, $i = 1, \ldots, m$, having only the one point $x_0$ in common and such that each $\alpha_i$ is contained in $X$ and separates one boundary component of $Y_i'$ from the other in $X$ but fails to separate the boundary components of each $Y_j', j \neq i$, in $X$, and each $\beta_i$ crosses the handle in $X'$ formed by $Y_i'$ once and otherwise $\beta_i \cap Y_j' = \varnothing$, $i \neq j$. Since $X$ is planar, it follows that $[y] =$

$\Pi_{i=1}^{m}[\alpha_i][\beta_i][\alpha_i]^{-1}[\beta_i]^{-1}$ in $\Pi_1(X', x_0)$. Now, in $(X', x_0)$, each loop $\alpha_i$ is homotopic to a loop $c_i'y'(c_i')^{-1}$ where $c_i'$ is a path from $x_0$ to one boundary component of $Y_i'$ and $y'$ is a loop which goes once around this boundary component of $Y_i'$. In $(M, x_0)$, $c_i'y'(c_i')^{-1}$ is homotopic to a $c_i y c_i^{-1}$ where $c_i$ is a loop based at $x_0$. Since $[y] = \Pi_{i=1}^{m}[\alpha_i][\beta_i][\alpha_i]^{-1}[\beta_i]^{-1}$ in $\Pi_1(X', x_0)$ and each $[\alpha_i] = [c_i][y][c_i]^{-1}$ in $\Pi_1(M, x_0)$, we have

$$(1) \qquad [y] = \prod_{i=1}^{m} [c_i][y][c_i]^{-1}[\beta_i][c_i][y]^{-1}[c_i]^{-1}[\beta_i]^{-1}.$$

The product (1) implies $[y]$ is in the second term of the lower central series (the commutator subgroup) of $\Pi_1(M, x_0)$. Since the terms in the lower central series are normal subgroups, $[c_i][y][c_i]^{-1}$ is also in the second term of the lower central series. Hence (1) implies $[y]$ is in the third term of the lower central series. Proceeding in this manner we see that $[y] \in (\Pi_1(M))_\omega$.

THEOREM 1. *If $K$ is of genus 1 and $l \notin (\Pi_1(C))_\omega$, then $l(K) \neq (S^1 \times S^2)'$.*

PROOF. Suppose $l(K) = (S^1 \times S^2)'$. By Lemma 1, there exists a planar surface $D$ properly embedded in $K$ such that the number of components of Bd $D$ is odd, each component of Bd $D$ is parallel to $K$'s longitude $l$ and $D$ is incompressible in $K$. Put $D$ in general position relative to the minimal spanning surface $S$ of genus 1. If any simple closed curve $x$ of $S \cap D$ bounds a disk in $S$, then, since $D$ is incompressible, $x$ bounds a disk in $D$. Suppose $x$ is innermost relative to $S$, i.e., the disk $x$ bounds in $S$ contains no points of $D$ in its interior. Then we may replace the disk $x$ bounds in $D$ by the one it bounds in $S$ and, pushing the new $D$ off $S$, we obtain a $D \cap S$ with fewer components. Since the above statements are true for $S$ interchanged with $D$, we may suppose that no simple closed curve of $S \cap D$ bounds a disk in either $S$ or $D$. If $S \cap D = \varnothing$, then $D \subset C$ and, by Lemma 2, $l \in (\Pi_1(C))_\omega$, contradicting our hypothesis. If $x$ is a simple closed curve of $S \cap D$ which separates $S$, then, since $S$ is of genus 1, $x$ and Bd $S$ cobound an annulus $Z$ in $S$. We may suppose Int $Z \cap D = \varnothing$. We then obtain two planar surfaces $D'$, $D''$ by adding two copies of $Z$ to the two components of $D - x$, respectively, and adjusting by pushing both off $Z$. Now one of the resulting two surfaces $D'$, $D''$, say $D'$, has an odd number of boundary components and $D' \cap S$ has fewer components than $D \cap S$. Suppose then that $x$ does not separate $S$ and that $x$ is innermost on $D$, i.e., one component $D'$ of $D - x$ contains no points of $S$. Let $N(x)$ be a small regular neighborhood of $x$ in $S$. Let $S' = S - \operatorname{Int} N(x)$ and form a new planar surface $D_0$ by adding two parallel copies of $D'$ to $S'$, i.e., fill in the two holes of $S'$ with copies of $D'$ to form $D_0$. Note that we may push $D_0$ off $S$ so that $D_0 \cap S = \varnothing$ and that $D_0$ has an odd number of boundary components. Now in all cases we have the contradiction that $l \in (\Pi_1(C))_\omega$ and the proof is complete.

If $F$ is a free group, then $F_\omega = 1$ [5, pp. 311–312]; thus, we have the following corollary.

COROLLARY 1. *If $K$ is of genus 1 and $C$ is a cube with two handles ($K$ has an algebraically unknotted, minimal spanning surface), then $l(K) \neq (S^1 \times S^2)'$.*

Corollary 1 implies that for every pretzel knot $K$ of genus 1 [7] we have $l(K) \neq (S^1 \times S^2)'$ (Theorem 1 of [6] does not apply to all pretzel knots since some have trivial Alexander polynomial.)

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