**N**₀⁻CATITORICITY OF PARTIALLY ORDERED SETS OF WIDTH 2

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**Abstract.** A result of J. Rosenstein is that every N₀-categorical theory of linear order is finitely axiomatizable. We extend this to the case of partially ordered sets of width 2.

In [4] J. Rosenstein proved that every N₀-categorical theory of linear order is finitely axiomatizable. Later in [6] we gave a proof of this fact utilizing the notion of a nuclear structure. Our method allowed us to extend Rosenstein's result to trees: we proved that every finite-branching N₀-categorical tree has a finitely axiomatizable theory, and that every N₀-categorical tree has a decidable theory. In this paper we again employ nuclear structures to extend Rosenstein's theorem in another way. Recall that a partially ordered set has width ≤ n if it has no antichain of length n + 1. Our main result, proved in §2, is that every N₀-categorical, partially ordered set of width 2 has a finitely axiomatizable theory.

Let us recall the definition of a nuclear structure, introduced in [6]. Suppose T is a complete theory. As usual, p is an n-type if it is a maximal set of formulas consistent with T, where the free variables in each formula are from the set {x₀, ..., xₙ₋₁}. If I ⊆ n, then let p|I be the set of formulas in p involving only the variables in {xᵢ: i ∈ I}. Now let A be a model of T, and suppose X = {a₀, ..., aₘ₋₁} ⊆ A, I = {i₀, ..., iₙ₋₁}, where i₀ < ... < iₙ₋₁ < m, Y = {aᵢ: i ∈ I} and a ∈ A. Then we say that Y is a nucleus of X for a if the following holds: if p is the (m + 1)-type realized by <a₀, ..., aₘ₋₁, a>, then p is the unique (m + 1)-type extending p|I U p|(I U {m}). We say that A is n-nuclear if for every finite X ⊆ A and a ∈ A, there is a nucleus Y of X for a such that |Y| < n. If A is n-nuclear for some n < ω, then A is nuclear.

The relevant fact about nuclear structures is that if A is N₀-categorical and nuclear, and the language of A is finite, then Th(A) is finitely axiomatizable. (See [6] for details.)

The cornerstone of any investigation into N₀-categoricity is the fundamental Ryll-Nardzewski Theorem [5]. This theorem asserts that a complete theory T

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is $\aleph_0$-categorical iff for each $n < \omega$ the number of its $n$-types is finite. We will use this theorem frequently.

1. **Monotone relations on linearly ordered sets.** Let $(A, <)$ be a linearly ordered set. A binary relation $R \subseteq A \times A$ is monotone iff the following two conditions hold:
   1. if $(x, y) \in R$ and $x' < x$, then $(x', y) \in R$;
   2. if $(x, y) \in R$ and $y < y'$, then $(x, y') \in R$.

The purpose of this section is to prove the following theorem.

**Theorem 1.** If $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1})$ is $\aleph_0$-categorical, where $(A, <)$ is a linearly ordered set and each of $R_0, \ldots, R_{n-1}$ is monotone, then $\text{Th}(\mathfrak{A})$ is finitely axiomatizable.

Before beginning the proof, let us note some things about monotone relations. Consider any linearly ordered set $(B, <)$. Notice that $<$ itself is monotone, as is $\emptyset$. If $R$ is monotone, then

$$R^* = \{(x, y) \in B \times B : (y, x) \notin R\}$$

is monotone. If $R$ and $S$ are both monotone, then their composition

$$RS = \{(x, y) \in B \times B : (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z \in B\}$$

is also monotone. If $R$ and $S$ are monotone, then $R^*$ is definable in $(B, <, R)$ and $RS$ is definable in $(B, <, R, S)$.

We will call a structure $\mathfrak{B} = (B, <, S_0, \ldots, S_{m-1})$ a monotone algebra iff each of the following holds:

1. $(B, <)$ is a linearly ordered set;
2. each $S_i$ is monotone;
3. some $S_i$ is $<$, and some $S_i = \emptyset$;
4. for each $i < m$ there is $j < m$ such that $S_i^* = S_j$;
5. for each $i, j < m$, there is $k < m$ such that $S_i S_j = S_k$.

Consider again the $\aleph_0$-categorical structure $\mathfrak{A}$. By Ryll-Nardzewski's Theorem there are only finitely many monotone relations definable in $\mathfrak{A}$. Thus, without loss of generality, we can assume in Theorem 1 that $\mathfrak{A}$ is in fact a monotone algebra.

For monotone $R$ we let $R(x) = \{y : (x, y) \in R\}$.

**Lemma 1.1.** Suppose that $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1})$ is a monotone algebra, and that $a \in A$ and $y \in X \subseteq A$. Furthermore, suppose that there is an $i < n$ such that $a \in R_k(x)$ for any $x \in X$ and $k < n$,

(i) $a \in R_k(x) \iff R_i(y) \subseteq R_k(x)$.

Then, for each $k$, $r < n$ and $x \in X$,

(ii) $R_r(a) \subseteq R_k(x) \iff R_i R_r(y) \subseteq R_k(x)$.

**Proof.** Let $r < n$, and let $p$ be such that $R_p = R_i R_r$. Since $a \in R_i(y)$ it is clear that $R_r(a) \subseteq R_p(y)$. Now suppose that $s < n$ and $w \in X$ are such that
\[ R_r(a) \subseteq R_j(w) \subseteq R_p(y) \]. Let \( R_j = (R_rR^*_r)^* \). It is easy to check that \( z \in R_j(w) \) iff \( R_r(z) \subseteq R_j(w) \), and hence \( a \in R_j(w) \). Thus (i) implies that \( R_j(y) \subseteq R_j(w) \), so that whenever \( z \in R_j(y) \), then \( R_r(z) \subseteq R_j(w) \). But this implies that \( R_p(y) \subseteq R_j(w) \), from which it follows that \( R_p(y) = R_j(y) \). \[ \square \]

There is a dual form of this lemma.

**Lemma 1.2.** Suppose that \( \mathcal{A} = (A, <, R_0, \ldots, R_{n-1}) \) is a monotone algebra, and that \( a \in A \) and \( z \in X \subseteq A \). Furthermore, suppose that there is an \( i < n \) such that for any \( x \in X \) and \( k < n \),

(i) \( a \in R_k(x) \iff R_k(x) \subseteq R_i(z) \).

Then, for each \( k, r < n \) and \( x \in X \),

(ii) \( R_k(x) \subseteq R_r(a) \iff R_k(x) \subseteq (R^*_r R^*_r)^*(z) \).

**Proof.** Consider the "dual" monotone algebra \( \mathcal{A}' = (A, >, R'_0, \ldots, R'_{n-1}) \), where \( (u, v) \in R'_k \) iff \( (u, v) \notin R_k \). Thus, whenever \( x \in X \) and \( k < n \), then \( a \in R'_k(x) \iff R'_k(z) \subseteq R_k(x) \). Applying Lemma 1.1, \( R'_r(a) \subseteq R'_k(x) \iff R'_r R'_r(z) \subseteq R'_k(x) \), so that \( R'_k(x) \subseteq R'_r(a) \iff R'_k(x) \subseteq (R'_r R'_r)^*(z) \). Finally, as is easily checked, note that \( (R'_r R'_r)^* = (R^*_r R^*_r)^* \). \( \square \)

**Remark.** The subset \( R_i R_j(y) \), mentioned in Lemma 1.1, is definable in \( (\mathcal{A}, R_i(y)) \). Similarly, the subset \( (R^*_i R^*_r)^*(z) \), mentioned in Lemma 1.2, is definable in \( (\mathcal{A}, R_i(z)) \).

Notice that whenever \( a \in A \) and \( X \) is a nonempty finite subset of \( A \), then there are \( y, z \in X \) which do satisfy the hypotheses of Lemmas 1.1 and 1.2, respectively. We will refer to such a subset \( \{y, z\} \) of \( X \) as a prenucleus of \( X \) for \( a \).

**Lemma 1.3.** If \( \mathcal{A} = (A, <, R_0, \ldots, R_{n-1}) \) is an \( \aleph_0 \)-categorical monotone algebra, then \( \mathcal{A} \) is 2-nuclear. In fact, if \( a \in A \) and \( X \subseteq A \) is nonempty and finite, then any prenucleus of \( X \) for \( a \) is also a nucleus of \( X \) for \( a \).

**Proof.** We can suppose that \( \mathcal{A} \) is countable. Consider a finite sequence \( \langle x_0, \ldots, x_m \rangle \) of elements from \( A \). Let us say that \( R_p(x_i) \) and \( R_q(x_i) \) are neighbors (with respect to \( \langle x_0, \ldots, x_m \rangle \)) if whenever \( k \leq m \) and \( r < n \) are such that either \( R_p(x_i) \subseteq R_r(x_k) \subseteq R_q(x_j) \) or \( R_q(x_j) \subseteq R_r(x_k) \subseteq R_p(x_i) \), then either \( R_r(x_k) = R_p(x_i) \) or \( R_r(x_k) = R_q(x_j) \). Let us say that \( \langle x_0, \ldots, x_m \rangle \) and \( \langle y_0, \ldots, y_m \rangle \) are equivalent iff whenever \( R_p(x_i) \) and \( R_q(x_j) \) are neighbors then \( \langle \mathcal{A}, R_p(x_i), R_q(x_j) \rangle \equiv \langle \mathcal{A}, R_p(y_i), R_q(y_j) \rangle \). Easily, if \( \langle x_0, \ldots, x_m \rangle \) and \( \langle y_0, \ldots, y_m \rangle \) are equivalent then they satisfy the same quantifier-free formulas in \( \mathcal{A} \), and \( R_p(x_i) \) and \( R_q(x_j) \) are neighbors iff so are \( R_p(y_i) \) and \( R_q(y_j) \).

So suppose \( \langle x_0, \ldots, x_m \rangle \) and \( \langle y_0, \ldots, y_m \rangle \) are equivalent, and suppose that \( \{x_i, x_j\} \) is a prenucleus of \( \{x_0, \ldots, x_m\} \) for \( a \) as demonstrated by the neighbors \( R_p(x_i) \) and \( R_q(x_j) \). Thus there is some \( b \in A \) such that \( \langle \mathcal{A}, R_p(x_i), R_q(x_j), a \rangle \equiv \langle \mathcal{A}, R_p(y_i), R_q(y_j), b \rangle \). Now it follows from Lemmas 1.1 and 1.2 and the Remark that \( \langle x_0, \ldots, x_m, a \rangle \) and \( \langle y_0, \ldots, y_m, b \rangle \) are equivalent. Continuing in a back-and-forth manner, we can build an automorphism \( f: \mathcal{A} \to \mathcal{A} \) such that \( f(x_i) = y_i \) for \( i \leq m \) and \( f(a) = b \). Thus \( \langle x_0, \ldots, x_m, a \rangle \) and \( \langle y_0, \ldots, y_m, b \rangle \)
realize the same type, so that \( \{x_i, x_j\} \) is indeed a nucleus of \( \{x_0, \ldots, x_m\} \) for \( a \).

\( \square \)

This proves Theorem 1. We can easily get a slight strengthening of this theorem.

**Corollary 1.4.** If \( \mathfrak{A} = (A, <, R_0, \ldots, R_{n-1}, U_0, \ldots, U_{m-1}) \) is \( \aleph_0 \)-categorical, where \( (A, <) \) is a linearly ordered set, each \( R_i \) is monotone and each \( U_j \subseteq A \), then \( \text{Th}(\mathfrak{A}) \) is finitely axiomatizable.

**Proof.** For each \( j < m \), let

\[
S_j = \{(x, y) \in A \times A : x \leq y, \text{and if } y \in U_j, \text{then } x \neq y\}.
\]

Then each \( S_j \) is a monotone relation which is definable in \( \mathfrak{A} \), and each \( U_j \) is definable in \( (A, <, R_0, \ldots, R_{n-1}, S_0, \ldots, S_{m-1}) \). Apply Theorem 1. \( \square \)

2. Partially ordered sets of width 2. In this section we prove the main result.

**Theorem 2.** If \( \mathfrak{A} = (A, <) \) is an \( \aleph_0 \)-categorical, partially ordered set of width 2, then \( \text{Th}(\mathfrak{A}) \) is finitely axiomatizable.

First, we introduce some notation which will apply to any partially ordered set \( (B, <) \). If \( x, y \in B \), then let \( x \not< y \) denote that \( x \) and \( y \) are incomparable (i.e., neither \( x < y \) nor \( y < x \)). For \( k < \omega \), let \( E_k \) be the binary relation such that

\[
E_k(x, y) \iff \exists x_0, \ldots, x_k(x = x_0 \land \cdots \land x_k = y),
\]

and let \( E \) be such that \( E(x, y) \iff \exists k E_k(x, y) \). Notice that \( E \) is an equivalence relation on \( B \). Each \( E_k \) is definable in \( (B, <) \), but in general \( E \) is not. However, if \( (B, <) \) is \( \aleph_0 \)-categorical, then a consequence of Ryll-Nardzewski's Theorem is that there is some \( n \) such that \( E(x, y) \iff E_k(x, y) \) for some \( k \leq n \). Thus if \( (B, <) \) is \( \aleph_0 \)-categorical, then \( E \) is definable. We call the equivalence classes of \( E \) components, and say that \( (B, <) \) is simple if \( B \) itself is a component.

Now we prove the theorem in the special case that, in addition to the given hypotheses, \( \mathfrak{A} \) is simple.

Let \( a \in A \) and define

\[
A_0 = \{x \in A : \mathfrak{A} \models E_k(a, x) \text{ for some even } k \leq n\},
\]

\[
A_1 = \{x \in A : \mathfrak{A} \models E_k(a, x) \text{ for some odd } k \leq n\}.
\]

It is easy to check that \( A_0 \) and \( A_1 \) are linearly ordered subsets of \( A \) and that \( A_0 \cup A_1 = A \) and \( A_0 \cap A_1 = \emptyset \). Define \( < \) on \( A \) so that

\[
x < y \iff x < y \lor (x \not< y \land x \in A_0 \land y \in A_1),
\]

and for \( e = 0, 1 \) define the binary relation \( R_e \) so that

\[
R_e(x, y) \iff \forall x_1, y_1((x_1 \leq x \land y \leq y_1 \land x_1 \in A_e) \rightarrow x_1 < y_1).
\]
It is clear that $<$ linearly orders $A$, and that $R_0$ and $R_1$ are monotone relations (with respect to $(A, <)$). Each of $A_0, A_1, <, R_0$ and $R_1$ is definable in $(A, <, a)$. Conversely, the relation $<$ is definable in $(A, <, R_0, R_1, A_0, A_1)$ by

$$x < y \iff (x \in A_0 \land R_0(x, y)) \lor (x \in A_1 \land R_1(x, y))$$

$$\lor ((x \in A_0 \iff y \in A_0) \land x < y).$$

Now, since $\mathcal{A}$ is $\aleph_0$-categorical, so is $(A, <, a)$, and hence also is $(A, <, R_0, R_1, A_0, A_1)$. But by Corollary 1.4, $\text{Th}(A, <, R_0, R_1, A_0, A_1)$ is finitely axiomatizable; therefore, so is $\text{Th}(\mathcal{A})$. This proves the theorem for simple $\mathcal{A}$.

Now consider any arbitrary $\aleph_0$-categorical $\mathcal{A}$ of width 2, and consider the relation $E$ on $A$. If $X$ is a component, then $\mathcal{A}|X$ is simple. It is easy to see that if $X, Y$ are different components and $x \in X, y \in Y$, then either $x < y$ or $y < x$. If, in addition, $x_1 \in X$ and $y_1 \in Y$, then $x_1 < y_1$ iff $x < y$. Thus, there is an induced linear order $<$ on the set of components. By Ryll-Nardzewski’s Theorem, there are components $X_0, \ldots, X_m$ such that if $Y$ is any component, then $\mathcal{A}|Y = \mathcal{A}|X_j$ for some $j < m$. By the first part of this proof, $\text{Th}(\mathcal{A}|X_j)$ is finitely axiomatizable for each $j < m$. Now let $B$ be the set of components, and let $U_j = \{Y \in B: Y = X_j\}$. Then $\mathcal{B} = (B, <, U_0, \ldots, U_m)$ is $\aleph_0$-categorical, so that by Rosenstein’s result [4] (or Corollary 1.4), $\text{Th}(\mathcal{B})$ is finitely axiomatizable. Now it is an easy matter to see how to recover $\mathcal{A}$ from the structures $\mathcal{A}|X_0, \ldots, \mathcal{A}|X_m$ and $\mathcal{B}$. Hence, it can be inferred on general grounds (e.g. Fefferman and Vaught [3]), or shown directly, that $\text{Th}(\mathcal{A})$ is finitely axiomatizable. □

3. Comments on the proof. An analysis of the proof of Theorem 2 reveals that $\text{Th}(\mathcal{A})$, for $\mathcal{A}$ an $\aleph_0$-categorical, partially ordered set of width 2, is 2-nuclear. To see this, let $E_x$ be the component of $A$ containing $x$, and let $<_x$ be the linear order on $E_x$ that $x$ induces. Now suppose that $a \in A$, and that $X \subseteq A$ is nonempty and finite. Then there is a $Y \subseteq X$ consisting of at most 2 elements such that

1. there is $y \in Y$ such that for all $x \in X \cap E_a$, if $x \leq_a a$, then $x \leq_a y$ $\leq_a a$;

2. there is $y \in Y$ such that for all $x \in X$, if $E_x \leq E_a$, then $E_x \leq E_y$ $\leq E_a$;

3. the “duals” of (1) and (2) are true.

Then $Y$ is a nucleus of $X$ for $a$.

Another observation concerning the proof. Suppose that $X_0, \ldots, X_m$ are components of $\mathcal{A}$ such that for any component $Y$ there is a unique $j \leq m$ such that $\mathcal{A}|Y = \mathcal{A}|X_j$. For each $j \leq m$, let $p_j$ be a 1-type realized by some element in $X_j$. For each component $Y$ let $a_Y \in Y$ be such that $\mathcal{A}|Y = \mathcal{A}|X_j$, then $a_Y$ realizes the type $p_j$. Let

$$A_0 = \{x \in A: \mathcal{A} \vDash E_k(a_Y, x) \text{ for some even } k \text{ and some component } Y\},$$

$$A_1 = \{x \in A: \mathcal{A} \vDash E_k(a_Y, x) \text{ for some odd } k \text{ and some component } Y\}.$$
Then, as before, \( A_0 \) and \( A_1 \) are linearly ordered subsets of \( A \) such that \( A_0 \cup A_1 = A \) and \( A_0 \cap A_1 = \emptyset \). It is not hard to check that, in addition, \((\mathbb{N}, A_0, A_1)\) is \( \mathbb{N}_0 \)-categorical.

The previous discussion recalls the fundamental theorem of Dilworth [2] concerning partially ordered sets of width \( n \). Dilworth’s Theorem asserts that if a partially ordered set \((A, <)\) has width \( n \), then \( A \) can be partitioned into \( n \) chains \( A_0, \ldots, A_{n-1} \). We have seen that in the case \( n = 2 \), if we start with an \( \mathbb{N}_0 \)-categorical partially ordered set, then we can find these chains so as to preserve \( \mathbb{N}_0 \)-categoricity. This suggests the following natural question.

**Question 3.1.** If \((A, <)\) is an \( \mathbb{N}_0 \)-categorical, partially ordered set of width \( n \), do there exist chains \( A_0, \ldots, A_{n-1} \) such that \( A = A_0 \cup \cdots \cup A_{n-1} \) and \((A, <, A_0, \ldots, A_{n-1})\) is \( \mathbb{N}_0 \)-categorical?

An affirmative answer to this question would imply that every \( \mathbb{N}_0 \)-categorical, partially ordered set of finite width has a decidable theory. This is a consequence of the following proposition.

**Proposition 3.2.** If \( \mathcal{U} = (A, <, A_0, \ldots, A_{n-1}) \) is an \( \mathbb{N}_0 \)-categorical structure such that \((A, <)\) is a partially ordered set, \( A_0, \ldots, A_{n-1} \) are chains, and \( A = A_0 \cup \cdots \cup A_{n-1} \), then \( \text{Th}(\mathcal{U}) \) is finitely axiomatizable.

**Proof.** We can assume that \( i < j < n \) implies that \( A_i \cap A_j = \emptyset \). Define \(<\) on \( A \) by

\[
  x < y \iff \exists i, j (x \in A_i \land y \in A_j \land (j \leq i \implies i = j \land x < y)).
\]

For \( i, j < n \) define \( R_{ij} \) by

\[
  R_{ij}(x, y) \iff \forall x_1, y_1 ((x_1 \leq x \land y \leq y_1 \land x_1 \in A_i \land y_1 \in A_j) \implies x_1 < y_1).
\]

Each \( R_{ij} \) is monotone with respect to \((A, <)\), so that by Corollary 1.4, \( \mathcal{U} = (A, <, A_0, \ldots, A_{n-1}, R_{ij}, i, j < n) \) has a finitely axiomatizable theory. But \(<\) is definable in \( \mathcal{U} \), so that \( \mathcal{U} \) also has a finitely axiomatizable theory. \( \square \)

4. Epilogue. Let \( T_n \) be the theory of partially ordered sets of width \( \leq n \). The theory \( T_2 \), in spite of Theorem 2, is quite a bit more complicated than the theory \( T_1 \) of linearly ordered sets. For, it can be shown that \( T_2 \) is undecidable, whereas, as is well known, \( T_1 \) is decidable. We will spare the reader the details, although it is not difficult to see how to encode into \( T_2 \) the print-out of any Turing machine. Then for any r.e. set \( X \), we can get a recursive sequence \( \sigma_0, \sigma_1, \sigma_2, \ldots \) of sentences such that for each \( n < \omega \), all of the following are equivalent:

1. \( n \in X \);
2. \( T_2 \cup \{ \sigma_n \} \) is complete;
3. \( T_2 \cup \{ \sigma_n \} \) has a finite model.

Since every finite model of \( T_2 \) is discrete, and no infinite discrete model of \( T_2 \) is \( \mathbb{N}_0 \)-categorical, we see that we can include a fourth condition equivalent to (1)–(3):

4. \( T_2 \cup \{ \sigma_n \} \) is \( \mathbb{N}_0 \)-categorical.
Thus, it follows that the set of $\aleph_0$-categorical sentences $\sigma$ consistent with $T_2$ is not recursive. This contrasts with the fact (see [1]) that the set of $\aleph_0$-categorical sentences consistent with $T_1$ is recursive. It is not clear whether or not the set of $\aleph_0$-categorical sentences consistent with $T_2$ is actually r.e. However, a consequence of the nuclearity is that the union of this set and the set of all sentences inconsistent with $T_2$ is r.e.

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