PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 2, April 1977

ORIENTABLE LINE-ELEMENT PARALLELIZABLE MANIFOLDS

R. MICHAEL ALLISTON

ABSTRACT. Examples of oriented, nonbounding, line-element parallelizable manifolds are given in all odd dimensions $n \neq 5$ or $2^r - 1$. Furthermore, these examples are indecomposable in the unoriented bordism ring, and hence represent generators of Tor Ω_*, the torsion subgroup of the oriented bordism ring. It is also proven that every class of Tor Ω_* admits a representative M such that $\tau(M) \oplus 2$ splits as a sum of line bundles over M.

1. **Introduction.** Massey and Szczarba [6] introduced the concept of a line-element parallelizable manifold, i.e. a manifold M whose tangent bundle $\tau(M)$ is a Whitney sum of line bundles. They constructed an oriented 4-manifold which is line-element parallelizable, but not parallelizable. However, their example, being a sphere bundle, bounds. Iberkleid [5] has exhibited many nonbounding line-element parallelizable manifolds, all of which are unorientable. Here we answer a question he posed by producing oriented line-element parallelizable manifolds which are indecomposable generators of the unoriented bordism ring $\mathfrak{N}_*$ in all odd dimensions not equal to 5 or $2^r - 1$.

We note that every oriented line-element parallelizable manifold belongs to Tor Ω_*, the torsion subgroup of the oriented bordism ring. Although we are unable to show that all classes in Tor Ω_* are so represented, we do show that each α in Tor Ω_* is represented by a manifold M with $\tau(M) \oplus 2$ being a sum of line bundles.

These results form a portion of the author's dissertation, University of Virginia, 1976.

2. **Examples.** For a vector bundle ξ over M, let $RP(\xi)$ denote the associated real projective space bundle. Let $RP(n_1, \ldots, n_t) = RP(\lambda_1 \oplus \cdots \oplus \lambda_t)$, where λ_i is the pullback over $RP(n_1) \times \cdots \times RP(n_t)$ of the canonical line bundle over $RP(n_i)$.

Following Iberkleid [5], for $n \neq 2^s - 1$, let X^n be given by:

(1) if $n = 4s + 2$, $s \geqslant 0$, then

Received by the editors August 20, 1976 and, in revised form, September 27, 1976.
AMS (MOS) subject classifications (1970). Primary 57D75; Secondary 57D25, 58A30.
Key words and phrases. Line-element parallelizable manifold.

$$X^n = RP\big(0, 0, 0, \underbrace{1, \ldots, 1}_{2s}\big),$$

(2) if $n = 4s$, $s \geqslant 1$, then

$$X^n = RP\big(0, \underbrace{1, \ldots, 1}_{2s}\big),$$

(3) let ζ be the canonical line bundle over $RP(0, 1)$; then

$$X^5 = RP(\zeta \oplus 3),$$

(4) if $n = 2^p(2q + 1) - 1$, $n \neq 5$, $p > 0$, $q > 0$, then

$$X^n = RP\big(0, \underbrace{1, \ldots, 1}_{2^p q - 1}, 2^p\big).$$

These manifolds are indecomposable in $\mathfrak{N}_*$, and Iberkleid [5] shows that

(a) $\tau(X^n)$ is a sum of line bundles if $n \neq 2$, and

(b) if $n \neq 5$, at least one of the summands in $\tau(X^n)$ is trivial, often more than one is trivial.

He also gives the following

LEMMA 2.1 [5]. *Let ξ be a vector bundle over the closed manifold M. If ξ and $\tau(M)$ both split as sums of line bundles over M such that $\tau(M)$ has n trivial summands, then $\tau(RP(\xi))$ splits as a sum of line bundles with $n - 1$ trivial summands.*

LEMMA 2.2. *If $n \neq 2$ or 5 and t is chosen so that $n + t$ is even, then $RP(\tau(X^n) \oplus t)$ is a closed, orientable, line-element parallelizable, $2n + t - 1$ dimensional manifold.*

PROOF. Line-element parallelizability follows immediately from Lemma 2.1 and properties (a) and (b). A formula of Borel and Hirzebruch [4, §23] shows that $w_1(\tau(RP(\tau(X^n) \oplus t))) = (n + t)w_1(\lambda)$, where λ is the canonical line bundle over $RP(\tau(X^n) \oplus t)$. □

PROPOSITION 2.3. *If $n = 2^r s \neq 2$, then $RP(\tau(X^n) \oplus 2^r)$ is a closed, oriented, line-element parallelizable, $2^{r+1}s + 2^r - 1$ dimensional manifold which is indecomposable in $\mathfrak{N}_*$.*

PROOF. By Lemma 2.2, it suffices to show that the characteristic number $s_{2n+2^r-1}[RP(\tau(X^n) \oplus 2^r)] \neq 0$, where s_i denotes the Stiefel-Whitney class detecting indecomposability. This follows at once from

LEMMA 2.4. *If M is a closed n-manifold, then*

$$s_{2n+m}\big[RP(\tau(M) \oplus m \oplus 1)\big] = \binom{n + m - 1}{n} s_n[M].$$

The proof of this result is highly technical and we postpone it to §4. □

3. **Representing** Tor Ω_*. Since every line-element parallelizable manifold has trivial rational Pontrjagin classes, any oriented such manifold represents a class in Tor Ω_*. The examples given in Proposition 2.3 are indecomposable in Tor Ω_* since they are so in $\mathfrak{N}_*$, but they do not suffice to generate Tor Ω_*.

Iberkleid [5, Theorem 2.2] shows that: Every manifold is unorientedly cobordant to a manifold M such that $\tau(M) \oplus 1$ splits as a sum of line bundles. Analogously, we have

PROPOSITION 3.1. *Every class in* Tor Ω_* *admits a representative M such that $\tau(M) \oplus 2$ splits as a sum of line bundles.*

PROOF. Anderson [1] shows that every class in Tor Ω_* can be represented as $RP(\det(\tau(N)) \oplus k)$, where $\det(\tau(N))$ is the orientation line bundle of N and k is odd. Furthermore, the class of $RP(\det(\tau(N)) \oplus k)$ in Ω_* depends only on the class of N in $\mathfrak{N}_*$. Iberkleid's result permits us to choose N so that $\tau(N) \oplus 1$ is a sum of line bundles.

Now $\tau(RP(\det(\tau(N)) \oplus k)) = p^*\tau(N) \oplus \theta$, where p is the projection and θ is the bundle along the fibers. Moreover, $\theta \oplus 1 = p^*(\det(\tau(N)) \oplus k) \otimes \lambda$, where λ is the canonical line bundle over $RP(\det(\tau(N)) \oplus k)$ [4, §32]. Thus, $\tau(RP(\det(\tau(N)) \oplus k)) \oplus 2$ is a sum of line bundles. $\square$

NOTE. By Lemma 2.1, $\tau(RP(\tau(X^n) \oplus 2^r))$ often has many trivial summands. Thus Proposition 3.1 allows many products in Tor Ω_* to be represented by line-element parallelizable manifolds.

4. **Proof of Lemma 2.4.** Let R be $RP(\tau(M) \oplus m \oplus 1)$, π the projection, and λ the canonical line bundle over R. If $c = w_1(\lambda)$, then $H^*(R; Z_2)$ is a free $H^*(M; Z_2)$ module on classes $1, c, \ldots, c^{n+m}$ with the single relation [4, §23]

$$c^{n+m+1} = c^{n+m}\pi^* w_1(M) + \cdots + c^{m+1}\pi^* w_n(M).$$

Now $\tau(R) = \pi^*\tau(M) \oplus \pi^*(\tau(M) \oplus m \oplus 1) \otimes \lambda$, [4, §32]. If we formally represent the Stiefel-Whitney classes of M as elementary symmetric functions on one dimensional classes $\alpha_1, \ldots, \alpha_n$, then we have

$$\text{(A)} \qquad s_{2n+m}(R) = \sum_{i=1}^{n} \alpha_i^{2n+m} + \sum_{i=1}^{n} (\alpha_i + c)^{2n+m} + (m+1)c^{2n+m}.$$

The first indexed summation of (A) is zero as each $\alpha_i^{2n+m} = 0$ for dimensional reasons. The final summand vanishes also, for Conner [3, Lemma 3.1] shows that

$$\text{(B)} \quad c^{n+m+a} = c^{n+m}\bar{w}_a(M) + \sum_{j=2}^{n+m+1} c^{n+m+1-j}\left(\sum_{i=0}^{a-1} \bar{w}_i(M) w_{a+j-i-1}(M)\right).$$

When $a = n$, any terms with $j > 1$ vanish for dimensional reasons, and as M immerses in R^{2n-1}, $\bar{w}_n(M) = 0$.

Examining the second indexed summation of (A), we have

$$\sum_{i=1}^{n} (\alpha_i + c)^{2n+m} = \sum_{a=0}^{2n+m} \binom{2n+m}{a} \left(\sum_{i=1}^{n} \alpha_i^{2n+m-a} \right) c^a$$

$$= \sum_{a=0}^{2n+m} \binom{2n+m}{a} s_{2n+m-a}(M) c^a.$$

For reasons of dimension, terms with $a < n + m$ vanish, and upon reindexing we have

$$\sum_{a=0}^{n} \binom{2n+m}{n-a} s_{n-a}(M) c^{n+m+a}.$$

Making the substitution at (B), all terms with $j > 1$ again vanish for dimensional reasons, and so (A) finally reduces to

$$\text{(C)} \qquad \sum_{a=0}^{n-1} \binom{2n+m}{n-a} \bar{w}_a(M) s_{n-a}(M) c^{n+m}.$$

Now

$$\langle w_a(M) s_{n-a}(M) c^{n+m}; [R] \rangle$$
$$= \langle \chi \mathrm{Sq}^a(s_{n-a}(M)); [M] \rangle \langle c^{n+m}; [RP(n+m)] \rangle,$$

where χ is the conjugation in the Steenrod algebra. Using a formula of Adams [2, Lemma 4],

$$\chi \mathrm{Sq}^a(s_{n-a}(M)) = \binom{2^q - n - 1}{a} s_n(M),$$

where q is arbitrarily large. Therefore evaluating (C) on $[R]$ gives

$$\sum_{a=0}^{n-1} \binom{2n+m}{n-a} \binom{2^q - n - 1}{a}.$$

This number reduces as follows. The coefficient of x^{n-a} in $(1 + x)^{2n+m}$ is

$$\binom{2n+m}{n-a},$$

and the coefficient of x^a in $(1 + x)^{2^q - n - 1}$ is

$$\binom{2^q - n - 1}{a}.$$

Thus the summation is equal to

$$\binom{2^q + n + m - 1}{n} - \binom{2n+m}{0} \binom{2^q - n - 1}{n},$$

which reduces modulo 2 to

$$\binom{n + m - 1}{n}. \quad \square$$

REFERENCES

1. Peter G. Anderson, *Cobordism classes of squares of orientable manifolds*, Ann. of Math. (2) **83** (1966), 47–53. MR **32** #4700.

2. J. F. Adams, *On formulae of Thom and Wu*, Proc. London Math. Soc. (3) **11** (1961), 741–752. MR **25** #2613.

3. P. E. Conner, *The bordism class of a bundle space*, Michigan Math. J. **14** (1967), 289–303. MR **37** #3579.

4. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Academic Press, New York; Springer-Verlag, Berlin, 1964. MR **31** #750.

5. W. Iberkleid, *Splitting the tangent bundle*, Trans. Amer. Math. Soc. **191** (1974), 53–59. MR **50** #1269.

6. W.S. Massey and R.H. Szczarba, *Line-element fields on manifolds*, Trans. Amer. Math. Soc. **104** (1962), 450–456. MR **25** #4548.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115