ORIENTABLE LINE-ELEMENT PARALLELIZABLE MANIFOLDS

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Abstract. Examples of oriented, nonbounding, line-element parallelizable manifolds are given in all odd dimensions \( n \neq 5 \) or \( 2^r - 1 \). Furthermore, these examples are indecomposable in the unoriented bordism ring, and hence represent generators of \( \text{Tor} \, \Omega_* \), the torsion subgroup of the oriented bordism ring. It is also proven that every class of \( \text{Tor} \, \Omega_* \) admits a representative \( M \) such that \( \tau(M) \oplus 2 \) splits as a sum of line bundles over \( M \).

1. Introduction. Massey and Szczarba [6] introduced the concept of a line-element parallelizable manifold, i.e. a manifold \( M \) whose tangent bundle \( \tau(M) \) is a Whitney sum of line bundles. They constructed an oriented 4-manifold which is line-element parallelizable, but not parallelizable. However, their example, being a sphere bundle, bounds. Iberkleid [5] has exhibited many nonbounding line-element parallelizable manifolds, all of which are unorientable. Here we answer a question he posed by producing oriented line-element parallelizable manifolds which are indecomposable generators of the unoriented bordism ring \( \Omega_* \) in all odd dimensions not equal to 5 or \( 2^r - 1 \).

We note that every oriented line-element parallelizable manifold belongs to \( \text{Tor} \, \Omega_* \), the torsion subgroup of the oriented bordism ring. Although we are unable to show that all classes in \( \text{Tor} \, \Omega_* \) are so represented, we do show that each \( \alpha \) in \( \text{Tor} \, \Omega_* \) is represented by a manifold \( M \) with \( \tau(M) \oplus 2 \) being a sum of line bundles.

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2. Examples. For a vector bundle \( \xi \) over \( M \), let \( RP(\xi) \) denote the associated real projective space bundle. Let \( RP(n_1, \ldots, n_i) = RP(\lambda_1 \oplus \cdots \oplus \lambda_i) \), where \( \lambda_i \) is the pullback over \( RP(n_1) \times \cdots \times RP(n_i) \) of the canonical line bundle over \( RP(n_i) \).

Following Iberkleid [5], for \( n \neq 2^r - 1 \), let \( X^n \) be given by:

(1) if \( n = 4s + 2, s \geq 0 \), then

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parallelizable manifolds

\[ X^n = RP\left(0, 0, 0, 1, \ldots, 1\right) \]

(2) if \( n = 4s, s > 1 \), then

\[ X^n = RP\left(0, 1, \ldots, 1\right) \]

(3) let \( \xi \) be the canonical line bundle over \( RP(0, 1) \); then

\[ X^5 = RP(\xi \oplus 3) \]

(4) if \( n = 2^p(2q + 1) - 1, n \neq 5, p > 0, q > 0 \), then

\[ X^n = RP\left(0, 1, \ldots, 1, 2^p\right) \]

These manifolds are indecomposable in \( \mathcal{R}_* \), and Iberkleid [5] shows that
(a) \( \tau(X^n) \) is a sum of line bundles if \( n \leq 2 \), and
(b) if \( n \leq 5 \), at least one of the summands in \( \tau(X^n) \) is trivial, often more than one is trivial.

He also gives the following

**Lemma 2.1 [5].** Let \( \xi \) be a vector bundle over the closed manifold \( M \). If \( \xi \) and \( \tau(M) \) both split as sums of line bundles over \( M \) such that \( \tau(M) \) has \( n \) trivial summands, then \( \tau(RP(\xi)) \) splits as a sum of line bundles with \( n - 1 \) trivial summands.

**Lemma 2.2.** If \( n \neq 2 \) or 5 and \( t \) is chosen so that \( n + t \) is even, then \( RP(\tau(X^n) \oplus t) \) is a closed, orientable, line-element parallelizable, \( 2n + t - 1 \) dimensional manifold.

**Proof.** Line-element parallelizability follows immediately from Lemma 2.1 and properties (a) and (b). A formula of Borel and Hirzebruch [4, §23] shows that \( w_1(\tau(RP(\tau(X^n) \oplus t))) = (n + t)w_1(\lambda) \), where \( \lambda \) is the canonical line bundle over \( RP(\tau(X^n) \oplus t) \). □

**Proposition 2.3.** If \( n = 2s \neq 2 \), then \( RP(\tau(X^n) \oplus 2^r) \) is a closed, oriented, line-element parallelizable, \( 2^{r+1}s + 2^r - 1 \) dimensional manifold which is indecomposable in \( \mathcal{R}_* \).

**Proof.** By Lemma 2.2, it suffices to show that the characteristic number
\[ s_{2n+2r-1}[RP(\tau(X^n) \oplus 2^r)] \neq 0, \]
where \( s_i \) denotes the Stiefel-Whitney class detecting indecomposability. This follows at once from

**Lemma 2.4.** If \( M \) is a closed \( n \)-manifold, then

\[ s_{2n+m}[RP(\tau(M) \oplus m \oplus 1)] = \left( n + m - 1 \right)s_n[M]. \]

The proof of this result is highly technical and we postpone it to §4. □
3. **Representing** $\text{Tor } \Omega_*$. Since every line-element parallelizable manifold has trivial rational Pontrjagin classes, any oriented such manifold represents a class in $\text{Tor } \Omega_*$. The examples given in Proposition 2.3 are indecomposable in $\text{Tor } \Omega_*$ since they are so in $\mathcal{Y}_*$, but they do not suffice to generate $\text{Tor } \Omega_*$. 

Iberkleid [5, Theorem 2.2] shows that: Every manifold is unorientedly cobordant to a manifold $M$ such that $\tau(M) \oplus 1$ splits as a sum of line bundles. Analogously, we have

**Proposition 3.1.** Every class in $\text{Tor } \Omega_*$ admits a representative $M$ such that $\tau(M) \oplus 2$ splits as a sum of line bundles.

**Proof.** Anderson [1] shows that every class in $\text{Tor } \Omega_*$ can be represented as $\text{RP}(\det(\tau(N)) \oplus k)$, where $\det(\tau(N))$ is the orientation line bundle of $N$ and $k$ is odd. Furthermore, the class of $\text{RP}(\det(\tau(N)) \oplus k)$ in $\Omega_*$ depends only on the class of $N$ in $\mathcal{Y}_*$. Iberkleid's result permits us to choose $N$ so that $\tau(N) \oplus 1$ is a sum of line bundles.

Now $\tau(\text{RP}(\det(\tau(N)) \oplus k)) = p^*\tau(N) \oplus \theta$, where $p$ is the projection and $\theta$ is the bundle along the fibers. Moreover, $\theta \oplus 1 = p^*(\det(\tau(N)) \oplus k) \otimes \lambda$, where $\lambda$ is the canonical line bundle over $\text{RP}(\det(\tau(N)) \oplus k)$ [4, §32]. Thus, $\tau(\text{RP}(\det(\tau(N)) \oplus k)) \oplus 2$ is a sum of line bundles. □

**Note.** By Lemma 2.1, $\tau(\text{RP}(\tau(X) \oplus 2'))$ often has many trivial summands. Thus Proposition 3.1 allows many products in $\text{Tor } \Omega_*$ to be represented by line-element parallelizable manifolds.

4. **Proof of Lemma 2.4.** Let $R$ be $\text{RP}(\tau(M) \oplus m \oplus 1)$, $\pi$ the projection, and $\lambda$ the canonical line bundle over $R$. If $c = w_i(\lambda)$, then $H^*(R; \mathbb{Z}_2)$ is a free $H^*(M; \mathbb{Z}_2)$ module on classes $1, c, \ldots, c^{n+m} \oplus w_n(M)$. With the single relation [4, §23]

$$c^{n+m+1} = c^{n+m}w_1(M) + \cdots + c^{m+1} \pi w_n(M).$$

Now $\tau(R) = \pi^*(\tau(M)) \oplus \pi^*(\tau(M) \oplus m \oplus 1) \otimes \lambda$, [4, §32]. If we formally represent the Stiefel-Whitney classes of $M$ as elementary symmetric functions on one dimensional classes $\alpha_1, \ldots, \alpha_n$, then we have

$$(A) \quad s_{2n+m}(R) = \sum_{i=1}^{n} \alpha_i^{2n+m} + \sum_{i=1}^{n} (\alpha_i + c)^{2n+m} + (m + 1)c^{2n+m}.$$ 

The first indexed summation of (A) is zero as each $\alpha_i^{2n+m} = 0$ for dimensional reasons. The final summand vanishes also, for Conner [3, Lemma 3.1] shows that

$$(B) \quad c^{n+m+a} = c^{n+m}w_a(M) + \sum_{j=2}^{n+m+1} c^{n+m+1-j} \left( \sum_{i=0}^{a-1} \bar{w}_i(M)w_{a+j-i-1}(M) \right).$$

When $a = n$, any terms with $j > 1$ vanish for dimensional reasons, and as $M$ immerses in $R^{2n-1}$, $\bar{w}_n(M) = 0$. 

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Examining the second indexed summation of (A), we have
\[
\sum_{i=1}^{n} (\alpha_i + c)^{2n+m} = \sum_{a=0}^{2n+m} \left( \sum_{i=1}^{n} \alpha_i^{2n+m-a} \right) c^a
\]
\[
= \sum_{a=0}^{2n+m} \left( \sum_{i=1}^{n} \alpha_i^{2n+m-a} \right) s_{2n+m-a}(M) c^a.
\]
For reasons of dimension, terms with \( a < n + m \) vanish, and upon reindexing we have
\[
\sum_{a=0}^{n} \left( \frac{2n + m}{n - a} \right) s_{n-a}(M) c^{n+m+a}.
\]
Making the substitution at (B), all terms with \( j > 1 \) again vanish for dimensional reasons, and so (A) finally reduces to
\[
(C) \sum_{a=0}^{n-1} \left( \frac{2n + m}{n - a} \right) w_a(M) s_{n-a}(M) c^{n+m}.\]

Now
\[
\langle w_a(M) s_{n-a}(M) c^{n+m}; [R] \rangle = \langle \chi \text{Sq}^a(s_{n-a}(M)); [M] \rangle \langle c^{n+m}; [RP(n + m)] \rangle,
\]
where \( \chi \) is the conjugation in the Steenrod algebra. Using a formula of Adams [2, Lemma 4],
\[
\chi \text{Sq}^a(s_{n-a}(M)) = \left( \frac{2^q - n - 1}{a} \right) s_n(M),
\]
where \( q \) is arbitrarily large. Therefore evaluating (C) on \([R]\) gives
\[
\sum_{a=0}^{n-1} \left( \frac{2n + m}{n - a} \right) \left( \frac{2^q - n - 1}{a} \right).
\]
This number reduces as follows. The coefficient of \( x^{n-a} \) in \((1 + x)^{2n+m}\) is
\[
\left( \frac{2n + m}{n - a} \right),
\]
and the coefficient of \( x^a \) in \((1 + x)^{2q-n-1}\) is
\[
\left( \frac{2^q - n - 1}{a} \right).
\]
Thus the summation is equal to
\[
\left( \frac{2^q + n + m - 1}{n} \right) - \left( \frac{2n + m}{0} \right) \left( \frac{2^q - n - 1}{n} \right),
\]
which reduces modulo 2 to
\[
\left( \frac{n + m - 1}{n} \right). \quad \Box
\]
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