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ON THE DUALITY BETWEEN SMOOTHABILITY AND DENTABILITY

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ABSTRACT. By renorming l_1 with an equivalent dual norm it is shown that smoothability of the unit ball of a conjugate Banach space E^* does not imply dentability of the unit ball of either E or E^{**}. It is also shown that the unit ball may be smoothable yet fail to be smooth at any point.

1. Introduction.

1.1. A nonempty subset K of a Banach space E is said to be *smoothable* if for every $\varepsilon > 0$ there is an f in E^* with $\|f\| = 1$ and a closed ball $B \subset E$ such that the following conditions are satisfied:

(i) $\sup f[B] < \sup f[K]$,

(ii) $\{x \in K: f(x) \leqslant \sup f[K] - \varepsilon\} \subset B$.

The notion of smoothability is originally due to Edelstein [2]. The definition given above is as reformulated by Kemp [3].

There is a formal duality between the notion of smoothability and the notion of dentability (see [2] and §2 below); and in [2] it was shown that the smoothability and dentability properties of c_0, l_1, and m parallel this formal duality. For example, the unit ball of l_1 is dentable but not smoothable while the unit ball of m or c_0 is smoothable but not dentable. In [3] Kemp has generalized this by showing that if the unit ball of any Banach space is dentable then the unit ball of the conjugate space must be smoothable.[2] At the same time Kemp asked whether smoothability of the unit ball of E^* implies dentability of the unit ball of E, and whether the unit ball of E is smoothable iff the unit ball of E^* is dentable.

In this note we provide an answer to these questions with the following

1.2. THEOREM. *There is a conjugate Banach space, E^*, whose unit ball is smoothable while neither the unit ball of E nor the unit ball of E^{**} is dentable.*

It was also asked in [3] whether smoothability of the unit ball of E implies that the norm of E is Gâteaux differentiable at some point. We shall show

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[2]This in turn has been extended by Sullivan [5], who has shown that if the unit ball of E is dentable then every bounded subset of E^* is smoothable.

1.3. THEOREM. *There exists a Banach space E whose unit ball is smoothable and whose norm is not Gâteaux differentiable at any point.*

2. **Definitions.** If A and B are two subsets of a Banach space E we denote by $A + B$ the set $\{a + b: a \in A, b \in B\}$. The unit ball $\{x: \|x\| \leqslant 1\}$ of E will be denoted U, so that $x + rU$ is the closed ball of radius r centered at x.

2.1. A nonempty subset K of a Banach space E is said to be *dentable* [4], if for every $\varepsilon > 0$ there is some x in K which is not contained in the closed convex hull of $K \setminus (x + \varepsilon U)$.

Although neither dentability nor smoothability of K implies that K is bounded, we are interested only in the situation where K is bounded. In this case Definitions 1.1 and 2.1 can be recast in such a manner that they become formally dual to each other. This can be seen by noting that a bounded set K is smoothable iff given any $\varepsilon > 0$, there exist a point x and a closed hyperplane π of E, and some positive ρ such that the following three statements hold:

$$\pi \cap K \neq \varnothing, \tag{2.2}$$

$$K \setminus (\pi + \varepsilon U) \subset x + \rho U, \tag{2.3}$$

$$\operatorname{dist}(\pi, x + \rho U) > 0, \tag{2.4}$$

where $\operatorname{dist}(A,B) = \inf\{\|a - b\|: a \in A, b \in B\}$.

The three statements that are formally dual to the above are obtained by interchanging the symbols π and x in (2.2), (2.3), and (2.4). It is not difficult to verify that the bounded set K is dentable iff the three formally dual statements hold.

For the remainder of this note, Γ denotes an arbitrary infinite set, α is a fixed element of Γ, and if x is a point in either $c_0(\Gamma)$, $l_1(\Gamma)$, or $m(\Gamma)$, x' denotes the point whose value at $\gamma \in \Gamma$ is given by

$$x'(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \neq \alpha, \\ 0 & \text{if } \gamma = \alpha. \end{cases} \tag{2.5}$$

On each of $c_0(\Gamma)$, $l_1(\Gamma)$, and $m(\Gamma)$, the usual norm will be denoted $\|\cdot\|$. On $c_0(\Gamma)$ we define a new norm, $|||\cdot|||$, by specifying that its closed unit ball be V, where

$$V = \operatorname{Cl}(U + W), \tag{2.6}$$

and where

$$U = \{x \in c_0(\Gamma): \|x\| \leqslant 1\}, \quad W = \{x \in c_0(\Gamma): (x(\alpha))^2 + \|x'\|^2 \leqslant 1\}.$$

(By $\operatorname{cl}(U + W)$ we mean the closure of the subset $U + W$. For U and W as above, $\operatorname{cl}(U + W) = U + W$; but this fact is not required in the sequel, and the proof is therefore omitted.)

In §3 we will show that if $E = (c_0(\Gamma), |||\cdot|||)$, then E, E^*, and E^{**} have the properties stated in Theorem 1.2.

3. **Proof of Theorem 1.2.** To prove Theorem 1.2, we will first identify the spaces E^* and E^{**}, and then show that E, E^*, and E^{**} have the desired properties.

3.1. PROPOSITION. *Let* $E = (c_0(\Gamma), |||\cdot|||)$. *Then*:

(i) E^* *is isometric to* $(l_1(\Gamma), |||\cdot|||)$ *where* $|||\cdot|||$ *is defined on* $l_1(\Gamma)$ *by*

$$|||x||| = \|x\| + \left[(x(\alpha))^2 + \|x'\|^2\right]^{1/2}.$$

(ii) E^{**} *is isometric to* $(m(\Gamma), |||\cdot|||)$ *where* $|||\cdot|||$ *is the norm on* $m(\Gamma)$ *whose closed unit ball is* V^{**} *where*

$$V^{**} = U^{**} + W^{**}, \qquad U^{**} = \{x \in m(\Gamma) : \|x\| \leqslant 1\},$$

$$W^{**} = \left\{x \in m(\Gamma) : (x(\alpha))^2 + \|x'\|^2 \leqslant 1\right\}.$$

PROOF. On each of $c_0(\Gamma)$, $l_1(\Gamma)$, and $m(\Gamma)$ define an auxiliary norm $||||\cdot||||$ by

$$||||x|||| = \left[(x(\alpha))^2 + \|x'\|^2\right]^{1/2}.$$

For x in $c_0(\Gamma)$ we have $\|x\| \leqslant ||||x|||| \leqslant 2\|x\|$, so that on $c_0(\Gamma)$ the norm $||||\cdot||||$ is equivalent to $\|\cdot\|$. In particular, this means that the collection of continuous linear functionals on $(c_0(\Gamma), ||||\cdot||||)$ may be identified as being $l_1(\Gamma)$. We now show that $(l_1(\Gamma), ||||\cdot||||)$ is isometric to $(c_0(\Gamma), ||||\cdot||||)^*$. For x in $l_1(\Gamma)$ we have

$$\begin{aligned}
&\sup\left\{x(y) : ||||y|||| \leqslant 1, y \in c_0(\Gamma)\right\} \\
&\quad = \sup\left\{x(\alpha)\cdot y(\alpha) + x'(y') : [y(\alpha)]^2 + \|y\|^2 \leqslant 1\right\} \\
&\quad = \sup\left\{\sup\left\{\beta x(\alpha) + x'(y') : \beta^2 + \|y'\|^2 \leqslant 1\right\} : -1 \leqslant \beta \leqslant 1\right\}. \\
&\quad = \sup\left\{\beta x(\alpha) + (1-\beta^2)^{1/2}\|x'\| : -1 \leqslant \beta \leqslant 1\right\} \\
&\quad = \left[(x(\alpha))^2 + \|x'\|^2\right]^{1/2}.
\end{aligned}$$

Thus, as claimed, $(l_1(\Gamma), ||||\cdot||||)$ is isometric to $(c_0(\Gamma), ||||\cdot||||)^*$, and in a similar manner, $(m(\Gamma), ||||\cdot||||)$ may be shown to be isometric to

$$(l_1(\Gamma), ||||\cdot||||)^*.$$

The proof of assertion (i) now follows readily. Since U and W are, respectively, the closed unit balls of the isomorphic spaces $(c_0(\Gamma), \|\cdot\|)$ and $(c_0(\Gamma), ||||\cdot||||)$, it follows that $(c_0(\Gamma), |||\cdot|||)$ is isomorphic to $(c_0(\Gamma), \|\cdot\|)$. Again, the set of continuous linear functionals on $(c_0(\Gamma), |||\cdot|||)$ may be identified as being $l_1(\Gamma)$. For $x \in l_1(\Gamma)$, the continuity and linearity of x yields

$$\sup x[V] = \sup x[U] + \sup x[W] = \|x\| + ||||x||||,$$

that is, $\sup x[V] = |||x|||$, showing that $(c_0(\Gamma), |||\cdot|||)^*$ is isometric to $(l_1(\Gamma), |||\cdot|||)$.

To prove assertion (ii), we first note that the above results imply that $(l_1(\Gamma), |||\cdot|||)$ is isomorphic to $(l_1(\Gamma), \|\cdot\|)$, and so $m(\Gamma)$ is the set of continuous linear functionals on $(l_1(\Gamma), |||\cdot|||)$. Let $B \subset m(\Gamma)$ be the closed unit ball corresponding to the norm on $m(\Gamma)$ which is conjugate to the norm $|||\cdot|||$ in $l_1(\Gamma)$. To show that $B = V^{**}$, note that since both B and V^{**} are weak* compact, it suffices to show that $\sup x[B] = \sup x[V^{**}]$ for every $x \in l_1(\Gamma)$. By definition of B we have $\sup x[B] = |||x|||$, and by definition of U^{**} and W^{**} we have

$$|||x||| = \|x\| + ||||x|||| = \sup x[U^{**}] + \sup x[W^{**}] = \sup x[U^{**} + W^{**}],$$

which completes the proof of the theorem.

3.2. THEOREM. *The unit ball of* $(l_1(\Gamma), |||\cdot|||)$ *is smoothable while neither the unit ball of* $(c_0(\Gamma), |||\cdot|||)$ *nor the unit ball of* $(m(\Gamma), |||\cdot|||)$ *is dentable.*

PROOF. Since the unit ball U of $(c_0(\Gamma), \|\cdot\|)$ is not dentable [4], it follows that $U + W$, and thus V is not dentable, where U, V, W are as defined in (2.6). Similarly, since the unit ball U^{**} of $(m(\Gamma), \|\cdot\|)$ is not dentable [1], it follows that the unit ball V^{**} of $(m(\Gamma), |||\cdot|||)$ is not dentable.

To prove that the unit ball V^* of $(l_1(\Gamma), |||\cdot|||)$ is smoothable, let $\varepsilon > 0$ be given, let z be the continuous linear functional on $l_1(\Gamma)$ whose value at each point x is given by $z(x) = 2x(\alpha)$ and let $p \in l_1(\Gamma)$ be the point for which $p(\alpha) = -\frac{1}{2}$ and $p(\gamma) = 0$ for $\gamma \neq \alpha$.

The point p is in the unit ball V^*, and $\sup z(V^*) = z(-p) = 1$. To show that V^* is smoothable, it suffices to show that there is some closed ball $B = p + \rho V^*$ of radius $\rho < 2$ which contains $\{x \in V^*\colon z(x) \leqslant 1 - \varepsilon\}$, for then we would have $\sup z[B] = \rho - 1 < 1 = \sup z[V^*]$. In other words, it suffices to show that there is some positive $\rho < 2$ such that each point x of $\{x \in V^*\colon |||x||| \leqslant 1,\ x(\alpha) \leqslant \frac{1}{2}(1 - \varepsilon)\}$ is at distance less than ρ from p (where, of course, distance is measured using the norm $|||\cdot|||$).

Since $|||x||| = |x(\alpha)| + \|x'\| + (|x(\alpha)|^2 + \|x'\|^2)^{1/2}$, if $|||x||| \leqslant 1$ we find that

$$\|x'\| \leqslant \big(1 - 2|x(\alpha)|\big)/\big(2 - 2|x(\alpha)|\big). \tag{3.4}$$

The definition of p implies that $(x - p)' = x'$ for all x, and if $|||x||| \leqslant 1$ it now follows from (3.4) that $|||x - p||| \leqslant f(x(\alpha))$, where

$$f(x(\alpha)) = \left|x(\alpha) + \frac{1}{2}\right| + \frac{1 - 2|x(\alpha)|}{2 - 2|x(\alpha)|}$$
$$+ \left[\left|x(\alpha) + \frac{1}{2}\right|^2 + \left(\frac{1 - 2|x(\alpha)|}{2 - 2|x(\alpha)|}\right)^2\right]^{1/2}.$$

Since $|||x||| \leqslant 1$ and $x(\alpha) \leqslant \frac{1}{2}(1 - \varepsilon)$ together imply that $-\frac{1}{2} \leqslant x(\alpha) \leqslant \frac{1}{2}(1 - \varepsilon)$, the proof of the theorem will be completed once it is shown that $\sup\{f(x(\alpha)): -\frac{1}{2} \leqslant x(\alpha) \leqslant \frac{1}{2}(1 - \varepsilon)\} < 2$. Due to the compactness of the interval $[-\frac{1}{2}, \frac{1}{2}(1 - \varepsilon)]$ and the continuity of f on this interval, we need only show that $f(x(\alpha)) < 2$ for each $x(\alpha)$ in $[-\frac{1}{2}, \frac{1}{2}(1 - \varepsilon)]$.

Both $|x(\alpha) + \frac{1}{2}|$ and $(1 - 2|x(\alpha)|)/(2 - 2|x(\alpha)|)$ are increasing on $-\frac{1}{2} \leqslant x(\alpha) \leqslant 0$, and so f is also increasing on that interval. It follows that $f(x(\alpha)) < 2$ for each $x(\alpha)$ in $[-\frac{1}{2}, 0]$. To show that $f(x(\alpha)) < 2$ for $x(\alpha)$ in $(0, \frac{1}{2}(1 - \varepsilon)]$, let $x(\alpha) = \frac{1}{2}(1 - \delta)$, where $\varepsilon \leqslant \delta < 1$. Then

$$f(x(\alpha)) = 1 - \frac{\delta}{2} + \frac{\delta}{1 + \delta} + \left[\left(1 - \frac{\delta}{2}\right)^2 + \left(\frac{\delta}{1 + \delta}\right)^2\right]^{1/2}$$

$$= 1 - \frac{\delta}{2} + \frac{\delta}{1 + \delta} + \left[\left(1 + \frac{\delta}{2} - \frac{\delta}{1 + \delta}\right)^2 - \frac{\delta^2}{1 + \delta}\right]^{1/2},$$

and since $\delta^2/(1 + \delta) > 0$, it is clear that $f(x(\alpha)) < 2$ whenever $0 < x(\alpha) \leqslant \frac{1}{2}(1 - \varepsilon)$. This completes the proof of Theorem 3.2.

Theorem 1.2 now follows from Theorems 3.1 and 3.2, and Theorem 1.3 will follow from

3.3. THEOREM. *If Γ is uncountable, the norm $|||\cdot|||$ on $l_1(\Gamma)$ is nowhere Gâteaux differentiable, that is, the unit ball V^* is not smooth at any point.*

PROOF. Let $x \in V^*$, $|||x||| = 1$. It is well known that for uncountable Γ, the usual unit ball, U^*, of $l_1(\Gamma)$ is not smooth at any point. In particular, let z_1 and z_2 be two linearly independent vectors in $m(\Gamma)$ such that for $i = 1,2$, $\|z_i\| = 1 = z_i(x/\|x\|)$, and let $z_0 \in m(\Gamma)$ be such that $||||z_0|||| = 1 = z_0(x/||||x||||)$. Then $(z_i + z_0)(x) = 1$, and so $|||z_i + z_0||| \geqslant 1$. On the other hand, $z_i + z_0 \in U^{**} + W^{**} = V^*$ so $|||z_i + z_0||| \leqslant 1$, showing that

$$|||z_i + z_0||| = 1 = (z_i + z_0)(x) \quad \text{for } i = 1,2,$$

so that V^* is not smooth at x.

We comment that Theorem 1.3 is a parallel of Proposition 1 of [1]. It was shown in [1] that c_0 $(= c_0(\omega))$ contains a symmetric closed and bounded convex body which is dentable, but which does not contain a single extreme point; i.e., c_0 can be renormed so that its closed unit ball is dentable but does not contain any exposed points.

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REFERENCES

1. M. Edelstein, *Concerning dentability*, Pacific J. Math. **46** (1973), 111–114. MR **48** #2730.

2. ______, *Smoothability versus dentability*, Comment. Math. Univ. Carolinae **14** (1973), 127–133. MR **47** #9243.

3. D. C. Kemp, *A note on smoothability in Banach spaces*, Math. Ann. **218** (1975), 211–217.

4. M. A. Rieffel, *Dentable subsets of Banach spaces, with applications to a Radon-Nikodým theorem*, Functional Analysis (Proc. Conf., Irvine, Calif., 1966), Academic Press, London; Thompson, Washington, D. C., 1967, pp. 71–77. MR **36** #5668.

5. F. Sullivan, *Dentability, smoothability and stronger properties in Banach spaces*, (preprint).

6. R. Anantharaman and J. H. M. Whitfield, *Smoothability Banach spaces*, Notices Amer. Math. Soc. **23** (1976), A-535. Abstract #737-46-3.

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