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AN ABSTRACT EXISTENCE THEOREM AT RESONANCE[1]

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ABSTRACT. By Schauder's fixed point theorem and alternative method (bifurcation theory) an abstract existence theorem at resonance for operational equations is proved which contains as particular cases rather different existence theorems for ordinary and partial differential equations as those of Lazer and Leach and of Landesman and Lazer.

Landesman and Lazer [8] proved, by the use of the alternative method, an existence theorem for selfadjoint elliptic Dirichlet nonlinear problems. Their theorem has since been extended, by using the alternative method or other techniques, by various authors. Detailed references may be seen in Cesari [1]–[3].

In the present paper, again in terms of the alternative method, we prove an abstract theorem, from which one can derive most of the extensions referred to above. This abstract theorem is proved here by Schauder's fixed point theorem. We restrict ourselves to the selfadjoint case here.

For extensions and variants of the abstract theorem, including nonselfadjoint cases, we refer to Cesari [4]–[6].

Let S be a real separable Hilbert space with inner product (x, y) and norm $\|x\| = (x, x)^{1/2}$. Let E: $\mathfrak{D}(E) \to S$ be a linear operator with domain $\mathfrak{D}(E) \subset S$ and finite dimensional null space $S_0 = \ker E$. Let P: $S \to S$ denote the projection operator with range S_0 and null space $S_1 = (I - P)S$. Furthermore we assume that S_1 is also the range of E. Then E: $\mathfrak{D}(E) \cap S_1 \to S_1$ is one-one and onto, and the partial inverse H: $S_1 \to \mathfrak{D}(E) \cap S$ exists as a linear operator. We assume that H is a linear bounded compact operator, and that E, P, H satisfy the usual axioms (h_1) $H(I - P)E = I - P$; (h_2) $EP = PE$; (h_3) $EH(I - P) = I - P$.

We have depicted here in an abstract way a situation which is rather typical for the large class of differential operators E which in [1] we denote as "the selfadjoint case", while we certainly have particularized the corresponding structure. We refer to [3], [4], [5] for much more general classes of operators and corresponding structures, in particular, the nonselfadjoint case, in the alternative method.

Let N: $S \to S$ be a continuous operator in S not necessarily linear. The equation

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$$Ex = Nx, \qquad x \in S, \tag{1}$$

is equivalent to the system of auxiliary and bifurcation equations

$$x = Px + H(I - P)Nx, \tag{2}$$

$$P(Ex - Nx) = 0. \tag{3}$$

We refer to [3] for details, and we note that here, having assumed $\mathfrak{R}(P) = S_0 = \ker E$, the bifurcation equation reduces to $PNx = 0$. Also, for $x^* = Px$, the auxiliary equation has the form $x = x^* + H(I - P)Nx$. Let L denote the norm of the linear operator $H(I - P)$: $S \to S$, or $L = \|H(I - P)\|$. We denote by (B) and (N_0) the following two assumptions:

(B) there is a constant J_0 such that $\|Nx\| \leqslant J_0$ for all $x \in S$;

(N_0) there is a constant $R_0 \geqslant 0$ such that for all $x \in S$, $x^* \in S_0$, with $\|x^*\| \geqslant R_0$, $Px = x^*$, $\|x - x^*\| \leqslant LJ_0$, we have $(Nx, x^*) \leqslant 0$ (or always $(Nx, x^*) \geqslant 0$).

Condition (B) is the usual situation considered in the theorems of Lazer and Leach [9], Landesman and Lazer [8], Williams [11], and in some work of DeFigueiredo [7] and others. Condition (N_0) can be shown to be implied (cf. [3]) by the specific hypotheses of the aforementioned theorems. Variants of conditions (N_0) are implied by the hypotheses of other analogous theorems (cf. [3]). Condition (N_0) was encountered by Kannan in his previous work (see [1] for references).

In [4] Cesari has considered the condition (N_ε) analogous to (N_0) where $(Nx, x^*) \leqslant -\varepsilon\|x^*\|$ is required (or $(Nx, x^*) \geqslant \varepsilon\|x^*\|$) for some $\varepsilon > 0$. This condition is equally implied by the specific hypotheses of the same aforementioned theorems. Cesari has shown in [4] that condition (N_ε) implies a stronger conclusion.

THEOREM. *Under hypotheses* (B) *and* (N_0), *the equation* $Ex = Nx$ *has at least one solution* $x \in S$ *with* $\|x\| \leqslant (R_0^2 + L^2J_0^2)^{1/2}$.

PROOF. Let us assume we have $(Nx, x^*) \leqslant 0$ in (N_0). Let $w = (w_1, \ldots, w_m)$ be an orthonormal base for the finite dimensional space $S_0 = \ker E = PS$. Then, for $x^* \in S_0$, we have $x^* = \sum_1^m c_i w_i$, or briefly $x^* = cw$, where $c = (c_1, \ldots, c_m) \in R^m$, $c_i = (x^*, w_i)$, $i = 1, \ldots, m$, and $\|x^*\| = |c| = (\sum_1^m c_i^2)^{1/2}$. Then system (2,3) with $Px = x^*$ can be written in the form $x = cw + H(I - P)Nx$, $(PNx, w) = 0$, where in the last relation (PNx, w) simply denotes the m-vector $[(PNx, w_i), i = 1, \ldots, m]$. Let

$$L = \|H(I - P)\| = \|H\|, \quad J = LJ_0,$$

and let R_1, R_2, R, ε be constants so chosen that

$$0 \leqslant R_0 \leqslant R_1 < R_2 < R, \qquad R_2 + J_0 \leqslant R, \qquad 0 < \varepsilon < R_1/2.$$

We shall consider the transformation T: $(x, c) \to (\bar{x}, \bar{c})$ defined by

$$T\colon \bar{x} = cw + H(I - P)Nx, \qquad \bar{c} = c + F(\bar{x}, c),$$
$$(x, c) \in \mathcal{C} = \left[(x, c) | x \in S, c \in R^m, \|x\| \leqslant R + J, |c| \leqslant R\right],$$

where $x^* = cw = \sum_1^m c_i w_i$, $\bar{x}^* = \bar{c}w = \bar{c}_1 w_1 + \cdots + \bar{c}_m w_m$, $c, \bar{c} \in R^m$, and $F(\bar{x}, c) = (F_1, \ldots, F_m)$ is explicitly given below. Note that

$$\bar{x}^* = x^* + F(\bar{x}, c)w = x^* + \sum_1^m F_i w_i.$$

Here, for $0 \leqslant |c| \leqslant R_1$ we take $F(\bar{x}, c) = (PN\bar{x}, w)$, or $\bar{x}^* = x^* + PN\bar{x}$. For $R_2 \leqslant |c| \leqslant R$, or $R_2 \leqslant \|x^*\| = |c| \leqslant R$, we take

$$F(\bar{x}, c) = \left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}(x^*, w), \quad \text{or}$$

$$\bar{x}^* = x^* + \left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}x^*.$$

For $R_1 \leqslant \|x^*\| = |c| \leqslant R_2$ we take

$$F(\bar{x}, c) = \lambda(PN\bar{x}, w) + (1 - \lambda)\left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}(x^*, w),$$

or

$$\bar{x}^* = x^* + \lambda PN\bar{x} + (1 - \lambda)\left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}x^*,$$

$$\lambda = (R_2 - R_1)^{-1}(R_2 - |c|), \qquad 0 \leqslant \lambda \leqslant 1.$$

We have $\bar{c} = c$, or $\bar{x}^* = x^*$, if and only if $F(\bar{x}, c) = 0$. For $|c| \leqslant R_1$ we have $F = 0$ if and only if $PN\bar{x} = 0$. For $R_2 \leqslant |c| \leqslant R$ we have

$$Fw = \left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}x^*$$

with $x^* \neq 0$ and $(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\| \leqslant -\varepsilon\|PN\bar{x}\|$, and again $F = 0$ if and only if $PN\bar{x} = 0$. For $R_1 < |c| < R_2$, hence $0 < \lambda < 1$, we have

$$(Fw, x^*) = \lambda(PN\bar{x}, x^*) + (1 - \lambda)\left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}\|x^*\|^2$$

where $(PN\bar{x}, x^*) \leqslant 0$, (cf. below), $(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\| \leqslant -\varepsilon\|PN\bar{x}\|$, and again $F = 0$ if and only if $PN\bar{x} = 0$. Thus, concerning the transformation $T|\mathcal{C}$, we have $(x, c) = T(x, c)$ if and only if $x = x^* + H(I - P)Nx$ and $PNx = 0$, that is, the fixed points (x, c) of T on $\mathcal{C}$ correspond exactly to the solutions of system (2,3), that is, to the solutions of the original equation $Ex = Nx$.

Let us prove that T maps $\mathcal{C}$ into itself. First, for $(x, c) \in \mathcal{C}$ we have $x^* = cw$, $\|x^*\| = \|cw\| = |c| \leqslant R$, and

$$\|\bar{x}\| \leqslant \|cw\| + \|H(I - P)Nx\| \leqslant |c| + LJ_0 \leqslant R + J.$$

Now we have $P\bar{x} = cw = x^*$,

$$\|\bar{x} - P\bar{x}\| = \|\bar{x} - x^*\| = \|H(I - P)Nx\| \leqslant LJ_0 = J.$$

Thus, $(PN\bar{x}, x^*) = (N\bar{x}, x^*) \leqslant 0$ for $\|x^*\| \geqslant R_0$, in particular, for $R_1 \leqslant |c| = \|x^*\| \leqslant R$.

For $|c| \leqslant R_1$ we have $\bar{c} = c + F(\bar{x}, c) = c + (PN\bar{x}, w)$, hence

$$|\bar{c}| \leqslant |c| + \|PN\bar{x}\| \leqslant R_1 + J_0 \leqslant R_2 + J_0 \leqslant R.$$

For $R_2 \leqslant |c| \leqslant R$ we have

$$\bar{x}^* = \left\{1 + \left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}\right\}x^* = \Lambda x^*,$$

where Λ is the number in braces, $(PN\bar{x}, x^*) \leqslant 0$, $|(PN\bar{x}, x^*)| \leqslant J_0\|x^*\|$, $\|PN\bar{x}\| \leqslant J_0$, $\|x^*\| \geqslant R_2 > R_1$, $\varepsilon/R_1 \leqslant \frac{1}{2}$, $\frac{1}{4} = 1 - \frac{1}{2} - \frac{1}{4} \leqslant \Lambda \leqslant 1$. Thus, $\bar{x}^*$ is on the segment between x^* and $x^*/4$, and $|\bar{c}| \leqslant |c| \leqslant R$.

For $R_1 < c < R_2$ we have $0 < \lambda < 1$,

$$\bar{x}^* = \lambda(PN\bar{x}) + \left\{1 + (1-\lambda)\left[(PN\bar{x}, x^*) - \varepsilon\|PN\bar{x}\|\right](2J_0\|x^*\|)^{-1}\right\}x^*,$$

and $\|\bar{x}^*\| \leqslant \|PN\bar{x}\| + \|\{\cdots\}x^*\| \leqslant J_0 + R_2 \leqslant R$. We have proved that $T\colon \mathcal{C} \to \mathcal{C}$.

Let us prove that T is compact. For this we consider any (bounded) sequence (x_k, c_k), $k = 1,2,\ldots,$ of points of $\mathcal{C}$. Then the sequence Nx_k is bounded, actually,

$$\|(I-P)Nx_k\| \leqslant \|Nx_k\| \leqslant J_0, \qquad \|y_k\| = \|H(I-P)Nx_k\| \leqslant LJ_0,$$

and since H is compact, there is a subsequence, say still $[k]$, so that y_k is convergent in S. Certainly $x_k^* = c_k w$, $F(\bar{x}_k, c_k) = d_k w$ are bounded sequences, actually $|c_k| \leqslant R$, $|d_k| \leqslant R$, both c_k and d_k in R^m, a finite dimensional space. Thus, we can extract the subsequence, say still $[k]$, in such a way that c_k, d_k are convergent in R^m, and then $\bar{x}_k = x_k^* + y_k$, $\bar{c}_k = c_k + d_k$ are convergent in S and R^m, respectively. We have proved that T is compact.

By Schauder's fixed point theorem, $T\colon \mathcal{C} \to \mathcal{C}$ has at least one fixed point $(x, c) = T(x, c)$. If $(Nx, x^*) \geqslant 0$, then the proof is the same provided we take $-(PN\bar{x}, x^*)$ instead of $(PN\bar{x}, x^*)$ in the definition of F.

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