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DIFFEOMORPHISMS OF 3-MANIFOLDS WHICH ARE HOMOTOPY EQUIVALENT TO S^1

L. S. HUSCH[1]

ABSTRACT. Let h be a diffeomorphism of a 3-manifold M which is homotopy equivalent to the 1-sphere. Suppose that the collection of positive iterates of h has compact closure in the space of smooth mappings of M into itself and suppose that this closed set generated by h is not a group. Necessary and sufficient conditions are given that another diffeomorphism g be topologically equivalent to h.

Let g and h be diffeomorphisms of a smooth manifold M onto itself; g and h are *topologically equivalent* if there exists a homeomorphism f of M onto itself such that $g = f^{-1}hf$. Let $\mathfrak{D}(M)$ be the space of smooth mappings of M into itself with the fine C^1-topology [9]. Let $\Gamma(h)$ be the closure of $\{h^i: i \geqslant 0\}$ in $\mathfrak{D}(M)$ and let $\Gamma_0(h) = \Gamma(h) - \{h^i: i \geqslant 0\}$. With respect to composition, $\Gamma(h)$ is a topological semigroup with the topology induced from $\mathfrak{D}(M)$. If $\Gamma(h)$ is compact and $\Gamma(h)$ is not contained in $\Gamma_0(h)$, then $\Gamma_0(h)$ is a topological group [11]. In this paper, we consider the problem of determining necessary and sufficient conditions in order that two diffeomorphisms h and g be topologically equivalent when $\Gamma(h)$ and $\Gamma(g)$ are compact. We shall not consider the case that $\Gamma(h) \subseteq \overline{\Gamma_0(h)}$ which reduces to the well-studied case of smooth actions of compact Lie groups on manifolds. Let π_h be the identity element of $\Gamma_0(h)$ and let $I(h) = \text{image } \pi_h$. It has been shown that $\pi_h: M \to I(h)$ is almost a "vector bundle" (cf. Proposition 2). Our main result is the following.

THEOREM 1. *Let M be an open 3-dimensional manifold which is homotopy equivalent to the 1-sphere S^1. Let h and g be diffeomorphisms of M onto itself such that $\Gamma(h)$ and $\Gamma(g)$ are compact, $\Gamma(h) \not\subset \overline{\Gamma_0(h)}$, and $\Gamma(g) \not\subset \overline{\Gamma_0(g)}$. h and g are topologically equivalent if and only if (1) there exists a homeomorphism k of M onto itself such that $k(I(h)) = I(g)$; (2) $h|I(h)$ is topologically equivalent to $k^{-1}gk|I(g)$; (3) if $(h|M - I(h))_*$ is the homomorphism of $H_*(M - I(h))$ induced by h, then $(h|M - I(h))_* = (k^{-1}gk|M - I(h))_*$.*

For arbitrary smooth manifolds, the problem of classifying the topological type of diffeomorphisms h when $h|I(h)$ is the identity was studied by the author [3]. If h is not smooth, then $I(h)$ could be a wildly embedded 1-sphere

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in M [5]. Since many of the techniques used in proving Theorem 1 are in [3], we will appeal often to this paper and only provide sketches of some proofs.

Let M and h be as in Theorem 1. From [3] and [4], we have the following propositions.

PROPOSITION 2. *π_h: $M \to I(h)$ is a smooth map of constant rank r, $0 \leqslant r \leqslant 3$; $I(h)$ is a smooth submanifold of M of dimension r whose inclusion into M is a homotopy equivalence. For each $x \in I(h)$, $\pi_h^{-1}(x)$ is homeomorphic to $\mathbf{R}^{3-r}$, $(3-r)$-dimensional Euclidean space.*

PROPOSITION 3. *Let ρ: $M - I(h) \to O(h)$ be the natural projection of $M - I(h)$ onto the orbit space of $h|M - I(h)$. $O(h)$ is a 3-dimensional manifold and ρ is a covering map.*

The author is unable to show, in general, that π_h: $M \to I(h)$ is a locally trivial vector bundle; however, this will turn out to be true in the particular case with which we are working.

PROPOSITION 4. *If $I(h)$ is compact, then π_h: $M \to I(h)$ is a locally trivial vector bundle.*

PROOF. Let α: $N \to I(h)$ be a normal vector bundle of $I(h)$ in M. Note that we can choose N such that $\alpha = \pi_h|N$ [7]. Pick a Riemannian metric for this bundle and let Σ be the sphere bundle of radius 1. There exists $p > 0$ such that $\Sigma \cap h^{ip}(\Sigma) = \varnothing$ for all integers i (see [6] or use Proposition 3). Note that if D is the disk bundle bounded by Σ, then $M = \bigcup_{i<0} h^{ip}(D)$. Let D_i denote the disk bundle of radius i. By using the uniqueness theorem of normal disk bundles, one can construct a homeomorphism ϕ: $N \to M$ such that $\phi(D_i) = h^{-ip}(D)$ for all positive integers i and such that $\pi_h\phi = \alpha$.

Henceforth, let σ: $\Sigma \to I(h)$ be the associated sphere bundle of π_h: $M \to I(h)$ if $I(h)$ is compact; if $I(h)$ is noncompact, choose Σ as in the proof of Proposition 4.

If $h|I(h)$ is the identity, then one can define a map π_h': $O(h) \to I(h)$ such that $\pi_h = \pi_h'\rho$; this construction was invaluable in [3].

Since $h|I(h) = h\pi_h|I(h)$ and since $h\pi_h \in \Gamma_0$, we can identify Γ_0 with the compact group "generated" by $h|I(h)$. Hence Γ_0 is a compact Lie group. From classical transformation group theory, we have the following.

PROPOSITION 5. *Γ_0 is periodic or is isomorphic to either S^1 or $S^1 \times Z_2$.*

PROPOSITION 6. *There exists an embedding λ: $\Sigma \to M - I(h)$ such that $\pi_h\lambda = \alpha$, $h^i\lambda(\Sigma) \cap \lambda(\Sigma) = \varnothing$, for all integers i, and λ is a homotopy equivalence.*

PROOF. *Case* 1. Γ_0 is periodic with period p. Let $\pi' = \lim_{i\to+\infty} h^{pi}$; there exists an induced map π'': $O(h^p) \to I(h)$ such that $\pi' = \pi''\rho'$ where ρ': $M - I(h) \to O(h^p)$ is the natural projection. Let $I_0(h)$ be the orbit space of

$h|I(h)$ and let ρ_0: $I(h) \to I_0(h)$ be the natural projection. If we let ρ'': $O(h^p) \to O(h)$ be the natural map, then we can find π''': $O(h) \to I_0(h)$ such that $\rho_0\pi'' = \pi'''\rho''$. It is not difficult to see that π''' is a smooth proper map with maximal rank at each point; hence π''' is a locally trivial fibration [1]. By considering the skeleta of some triangulation of $I_0(h)$, we can build a sphere bundle σ_0: $\Sigma_0 \to I_0(h)$ and an embedding λ_0: $\Sigma_0 \to O(h)$ such that $\pi'''\lambda_0 = \sigma_0$ and, for each $x \in I(h)$, λ_0: $\sigma_0^{-1}(\rho_0(x)) \to \pi'''^{-1}(\rho_0(x))$ can be lifted to a homotopy equivalence λ_0: $\sigma_0^{-1}(\rho_0(x)) \to \rho^{-1}(x) - \{x\}$; for details, see the proof of Theorem 4.3 in [3]. Let ρ^*: $\Sigma_1 \to \Sigma_0$ be the pullback of ρ_0: $I(h) \to I_0(h)$ using the map σ_0 and let σ_1: $\Sigma_1 \to I(h)$ be the natural map. Note that there exists an embedding λ_1: $\Sigma_1 \to M - I(h)$ such that $\pi_h\lambda_1 = \sigma_1$ and λ_1 is a homotopy equivalence. Note that Σ is bundle equivalent to Σ_1. λ_1 is the desired embedding.

By using the fibration π''' and [1], we can show the following.

PROPOSITION 7. *If Γ_0 is periodic, then π_h: $M \to I(h)$ is a locally trivial vector bundle.*

Case 2. Γ_0 is not periodic and dimension $I(h) = 2$. Let ρ_1: $I(h) \to I_1(h)$ be the natural projection of $I(h)$ onto the orbit space with respect to the action of Γ_0 restricted to $I(h)$. From classical transformation group theory, $I_1(h)$ is homeomorphic to either $\mathbf{R}$ or $[0, +\infty)$. In the latter case, we have one exceptional orbit which corresponds to 0.

Note that there exists a smooth proper map ρ_2: $O(h) \to I_1(h)$ such that $\rho_2\rho = \rho_1\pi_h$: $M - I(h) \to I_1(h)$. Note that ρ_2 has rank 1 at each point of $O(h)$ except at the points which correspond to the possible exceptional orbit. Hence ρ_2 is a locally trivial fibration except over the possible exceptional orbit. It is now easy to find an embedding of Σ into $O(h)$ whose lifting to $M - I(h)$ is the desired embedding.

PROPOSITION 8. *If Γ_0 is not periodic and* dimension $I(h) = 2$, *then π_h:$M \to I(h)$ is a locally trivial vector bundle.*

Case 3. Γ_0 is not periodic and dimension $I(h) = 1$. Note that in this case $O(h)$ is compact. By [12], there exists a surface $T \subseteq O(h)$ such that $i_*(\pi_1(T)) = \rho_*(\pi_1(M - I(h)))$. We may assume that T is smoothly embedded in $O(h)$.

Let $x \in I(h)$ and consider $\rho|\pi_h^{-1}(x) - \{x\}$. Since Γ_0 is not periodic, $\rho|\pi_h^{-1}(x) - \{x\}$ is a 1-1 immersion of $\pi_h^{-1}(x) - \{x\}$ into $O(h)$. By using the implicit function theorem, one can show that $O(h)$ is foliated by $\mathfrak{F} = \{\mathfrak{F}_x = \rho(\pi_h^{-1}(x) - \{x\})$: $x \in I(h)\}$. We may assume that T is transverse regular with respect to this foliation. If we intersect T with leaves of $\mathfrak{F}$, we get on T a family of curves with singular points. Since the inclusion of T can be lifted to $M - I(h)$, $T \cap \mathfrak{F}_z$ is compact for each z. We may assume that each leaf contains at most one singular point. The following procedure was motivated by S. P. Novikov's proof of Theorem 6.1 of [10]. There are two types of singular points: "central" and "saddle". Let x be a central singular point;

close to x, the simple closed curves of the intersection of T with the leaves of $\mathfrak{F}$ must be null-homotopic on their respective leaves. In fact, if τ is a component of $T \cap \mathfrak{F}_z$ which is a simple closed curve bounding a disk on T, then τ must be null-homotopic on $\mathfrak{F}_z$ since the immersion of each leaf induces a monomorphism on the fundamental group. There exists a curve ξ in $T \cap \mathfrak{F}_w$ for some w such that ξ contains a simple closed curve ξ_0 which is the boundary of a disk D on T such that D contains x in its interior and D contains only one other singular point y on its boundary. ξ_0 bounds a disk E on its leaf. Note that (int E) $\cap$ T is either empty or consists of a finite number of simple closed curves $\xi_1, \ldots, \xi_r$. ξ_i bounds a disk D_i on T; suppose that D_i is contained in no D_j for $i \neq j$. On a leaf $\mathfrak{F}_{w(i)}$ close to the leaf containing E, let ξ_i' be the component of the intersection of T with $\mathfrak{F}_{w(i)}$ which is close to ξ_i. ξ_i' bounds a disk E_i on $\mathfrak{F}_{w(i)}$. Replace T by $(T - D_i') \cup E_i$ where D_i' is the cell on T bounded by ξ_i'. Hence we may assume that $T \cap (\text{int } E) = \varnothing$; note that each D_i contains at least one central singular point. Let us refer to the E_i's as singular disks. We may assume that the E_i's contain no singular points. Fix some E_i; we can find η in $T \cap \mathfrak{F}_u$ such that η contains a simple closed curve η_0 which is the boundary of a disk G_i on T such that G_i contains E_i in its interior and G_i contains only one singular point on its boundary. Replace G_i by the disk $\mathfrak{F}_i$ which is contained on the leaf which contains $\eta_0 = \text{bdry } \mathfrak{F}_i$. We may have to do some alterations to T as above to remove other intersections of T with G_i. Note that the number of singular disks we obtain is never greater than the original number of central singular points. Hence, we may assume that the intersection of T with the leaves of $\mathfrak{F}$ contains no central singular points and each singular disk contains a saddle singular point in its boundary. Let D be a singular disk with saddle point x; note that D has a product neighborhood $N \times \mathbf{R}$ in $O(h)$ such that for each $t \in \mathbf{R}$, $N \times \{t\}$ lies on a leaf. Hence we can tilt D slightly along this product structure keeping x fixed so that x is no longer a singular point. Hence we may assume that the intersection of T with the leaves of $\mathfrak{F}$ contains no singular disks or central singular points.

The above cutting and pasting procedures do not change the homotopy class of the inclusion of T into $O(h)$. Hence the inclusion of T into $O(h)$ can be lifted to an embedding ϕ of T into $M - I(h)$; note that π_h: $\phi(T) \to I(h)$ is a smooth function whose critical points correspond to the image under ϕ of the singular points of the intersection of T with the leaves of $\mathfrak{F}$. Let $\tilde{\pi}_h$: $\widetilde{\phi(T)} \to \mathbf{R}$ be the induced map of the covering space of $\phi(T)$ which is the pullback of the universal covering of $I(h)$. Note that $\tilde{\pi}_h$ is a Morse function. If there is a singular point in the intersection of T with the leaves of $\mathfrak{F}$, then it follows from Morse theory [8] that some component of $\widetilde{\phi(T)}$ has infinite genus. But no covering of the torus or the Klein bottle has this property. Hence T meets the leaves of $\mathfrak{F}$ in simple closed curves. It is easily verified that if $x \in I(h)$, then $\phi(T) \cap \pi_h^{-1}(x)$ is connected. We leave to the reader the construction of the map λ from Σ onto the image of ϕ. This completes the proof of Proposition 6.

We now proceed as in [3] to finish the proof of Theorem 1. We will sketch a proof indicating some of the minor changes which are needed.

Choose a Riemannian metric for the bundle π_h: $M \to I(h)$ and let Σ_t denote the sphere bundle of those elements of M of norm t, $t \geqslant 0$. Let β: $\Sigma \times (0, +\infty) \to M - I(h)$ be a diffeomorphism such that $\beta(\Sigma \times \{t\}) = \Sigma_t$ and $\pi_h \beta(s, t) = \sigma(s)$. By Proposition 1.3 of [3] we can find a homeomorphism γ of M onto itself such that $\gamma(\lambda(\Sigma)) = \Sigma_2$, $\gamma(h\lambda(\Sigma)) = \Sigma_1$, $\pi_h\gamma = \pi_h$ and the induced homomorphism of $H_1(M - I(h))$ is the identity. Define δ: $\Sigma \to \Sigma$ by $\beta^{-1}\gamma h \gamma^{-1}\beta(x, 2) = (\delta(x), 1)$. Note that δ is a homeomorphism such that $\sigma\delta = h\sigma$. Define h_1 on M by

$$h_1(z) = \begin{cases} h(z) & \text{if } z \in I(h), \\ \beta(\delta(x), t/2) & \text{if } z = \beta(x, t). \end{cases}$$

Using the same proof as the proof of Proposition 1.6 of [3], we can show the following.

PROPOSITION 9. *h and h_1 are topologically equivalent.*

Suppose that k_0 is a homeomorphism of $I(h)$ onto itself; one can easily show that k_0 can be extended to a homeomorphism k_0' of M onto itself such that the induced homomorphisms on $H_*(M - I(h))$, $(h|M - I(h))_*$ and $(k_0'^{-1}hk_0'|M - I(h))_*$, are the same. Let k be the homeomorphism given in the hypotheses of Theorem 1. Hence, we may assume that $h|I(h) = k^{-1}gk|I(h)$. Let λ', γ', and δ' be the analogues of λ, γ and δ, respectively, for $k^{-1}gk$. Since γ_*: $H_*(M - I(h)) \to H_*(M - I(h))$ is the identity isomorphism, δ and δ' induce the same isomorphism on $H_*(\Sigma)$. If the dimension of $I(h)$ is 2, then, since $\sigma\delta = h\sigma$, this implies that $\delta = \delta'$ and the theorem follows from Proposition 9.

Suppose that the dimension of $I(h)$ is 1. Let us first consider the case that M is homeomorphic to $S^1 \times \mathbf{R}^2$. Choose simple closed curves $s_1, s_2 \subseteq \Sigma$ such that s_1 is a fibre and $\sigma|s_2$ is a homeomorphism. Note that $\delta_*([s_1]) = \pm[s_1]$ and $\delta_*([s_2]) = r[s_1] \pm [s_2]$ where r is an integer and $[s]$ denotes the homology class of s.

Suppose that h and $h|I(h)$ are orientation-preserving. Hence $\delta_*([s_1]) = [s_1]$ and $\delta_*([s_2]) = r[s_1] + [s_2]$; note that g and $g|I(g)$ are orientation-preserving. Let Δ be an orientation-preserving homeomorphism of Σ onto itself such that $\sigma\Delta = h^{-1}\sigma$ and $\delta_*([s_2]) = [s_2]$; note that $\sigma\delta\Delta = \sigma$. Let Aut^+S^1 be the space of orientation-preserving homeomorphisms of S^1 onto itself and let $\mathrm{Emb}(w, S^1)$ be the space of embeddings of a point $w \in S^1$; give both these spaces the compact-open topology. Consider R: $\mathrm{Aut}^+S^1 \to \mathrm{Emb}(w, S^1)$ which is defined by $R(\phi) = \phi(w)$; the induced homomorphism $\pi_1\mathrm{Aut}^+S^1 \to \pi_1\mathrm{Emb}(w, S^1)$ is an isomorphism [2]. $\pi_1\mathrm{Emb}(w, S^1)$ is isomorphic to the integers; let $R_\#$ be the composition of these isomorphisms. We may assume that $R_\#([\text{identity}]) = 1$. Note that an element of $\pi_1\mathrm{Aut}^+S^1$ may be represented by a homeomorphism f of Σ onto itself such that $\sigma f = \sigma$ and that

$f_*([s_2]) = R_\#([f])[s_1] + [s_2]$. Since $R_\#[\delta\Delta] = R_\#[\delta'\Delta]$, we can find an isotopy Φ'_t, $t \in [0, 1]$, of Σ onto itself such that $\Phi'_0 = \delta\Delta$, $\Phi'_1 = \delta'\Delta$ and $\sigma\Phi'_t = \sigma$ for all t. The remainder of the argument is given in the proof of Theorem 1.7 in [3]; we give a brief sketch. Define Φ: $\Sigma \times [1, 2] \to \Sigma \times [1, 2]$ by $\Phi(x, t) = (\Phi'_{t-1}\Delta^{-1}\delta'^{-1}(x), t)$ and define F: $\Sigma \times (0, +\infty) \to \Sigma \times (0, +\infty)$ by $F(x, t) = \beta^{-1}h_1^q\beta\Phi(\delta'^{-q}(x), 2^q t)$ where q is the unique integer such that $1 < 2^q t \leqslant 2$. F is a homeomorphism such that $\sigma p'_1 F = \sigma p'_1$ where p'_1 is the projection of $\Sigma \times (0, +\infty)$ onto Σ. One can show that $F^{-1}\beta^{-1}h_1\beta F(x, t) = (\delta'(x), t/2)$. If we let g_1 be the analogue of h_1 for $k^{-1}gk$ and define F' on M by

$$F'(z) = \begin{cases} z, & z \in I(h), \\ \beta F \beta^{-1}(z), & z \in I(h), \end{cases}$$

then $F'^{-1}h_1F' = g_1$ and the theorem follows from Proposition 9.

If h and $h|I(h)$ are not orientation-preserving then the only changes in the above proof are that one may have to consider $\mathrm{Aut}^{-1}S$, the space of orientation-reversing homeomorphisms of S^1 and/or one may need Δ to be orientation-reversing and, thus, $\Delta_*([s_2]) = -[s_2]$.

Now suppose that M is the twisted bundle over S^1. Choose $s_1, s_2 \subseteq \Sigma$ as above; note that $\delta_*([s_1]) = [s_1]$ and $\delta_*([s_2]) = r[s_1] \pm [s_2]$ where $r \in \mathbf{Z}_2$.

Let Aut Σ be the set of homeomorphisms f of Σ onto itself such that $\sigma f = \sigma$ and let Sect Σ be the set of sections of Σ; give both sets the compact-open topology. Fix a section $\xi \in$ Sect Σ; define T: Aut $\Sigma \to$ Sect Σ by $T(f) = f \circ \xi$. The proof of the following is straightforward.

PROPOSITION 10. *T induces a bijection from π_0(Aut Σ) onto π_0(Sect Σ). The natural map of π_0(Sect Σ) into the free homotopy classes of mappings of $I(h)$ into Σ, $[I(h), \Sigma]$, is one-to-one.*

But there exists a bijection from $[I(h), \Sigma]$ onto $H_1(\Sigma)$ which carries the image of π_0(Sect Σ) onto the torsion subgroup. We can now proceed as above to complete the proof of Theorem 1.

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DEPARTMENT OF MATHEMATICS UNIVERSITY OF TENNESSEE KNOXVILLE, TENNESSEE 37916