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GENERALIZED HEWITT-SAVAGE THEOREMS FOR STRICTLY STATIONARY PROCESSES

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ABSTRACT. Generalizations of the Hewitt-Savage zero-one law are proved for strictly stationary processes. This takes the form of statements concerning inclusion and equality relationships among certain sigma-fields related to the process.

1. Let $\{X_n, n \geqslant 1\}$ be a sequence of real-valued random variables on the probability space $(\mathfrak{R}^\infty, \mathfrak{B}^\infty, P)$ where $\mathfrak{R}^\infty$ and $\mathfrak{B}^\infty$ are the usual product space and product Borel σ-field, respectively. X_n may be considered coordinate variables [1]. For $\omega \in \mathfrak{R}^\infty$, we denote the kth coordinate of ω by "$(\omega)_k$", so that if $\omega = (x_1, x_2, \ldots)$, then $(\omega)_k = x_k$. Let N be a finite subset of the positive integers J, and let σ be a permutation of N. σ defines a function on $\mathfrak{R}^\infty$ onto itself by $(\sigma\omega)_k = (\omega)_{k\sigma}$, $k \geqslant 1$ ($k\sigma = k$, $k \notin N$). It is easy to see that σ is a measurable map, i.e., $\sigma^{-1}A \in \mathfrak{B}^\infty$ for each $A \in \mathfrak{B}^\infty$. Let Σ be the class of all permutations of every finite subset of J, that is, Σ is the class of all such σ as defined above. An event A is called *exchangeable* (or symmetrically dependent) if $\sigma^{-1}A = A$ for all $\sigma \in \Sigma$. Actually, we want a slightly larger class of exchangeable events, namely, those for which the relation $\sigma^{-1}A = A(P)$ holds. Here, and hereafter, such a notation means that the relation is understood to hold modulo P-null sets. We define σ on functions by $(\sigma f)(\omega) = f(\sigma\omega)$.

The *exchangeable σ-field* $\mathfrak{E}$ is the smallest σ-field containing the exchangeable events; the *tail* σ-field $\mathfrak{T}$ is the class of events expressible in terms of X_n, $X_{n+1}, \ldots$ for any arbitrarily large chosen n; the *invariant* σ-field $\mathfrak{I}$ is the class of events such that $T^{-1}A = A$, where T is the usual shift transformation $(T\omega)_k = (\omega)_{k+1}$. As with $\mathfrak{E}$, the definitions of $\mathfrak{T}$ and $\mathfrak{I}$ are to be understood modulo P.

The Hewitt-Savage 0-1 law [2] asserts that for independent, identically, distributed X_n, $\mathfrak{E}$ is trivial, that is, $\mathfrak{E}$ consists only of events of probability 0 or 1. In this note we prove generalizations of this result for strictly stationary processes (the process is *strictly stationary* if $P(T^{-1}A) = P(A)$ for all events A):

THEOREM 1. *If the X_n process is strictly stationary, and if the measures P_n (see* §3) *are uniformly absolutely continuous with respect to P, then,*

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(1) $$\mathfrak{S} \subset \mathfrak{I} \subset \mathfrak{T} \quad (P).$$

If, in addition, each $\sigma \in \Sigma$ is P nonsingular, i.e., $P(\sigma^{-1}A) = 0$ when $P(A) = 0$, there is equality above, that is,

(2) $$\mathfrak{S} = \mathfrak{I} = \mathfrak{T} \quad (P).$$

An easy example serves to show why (2) can be false if the nonsingularity hypothesis is not fulfilled. Consider the two points ω_1 and ω_2 given by $(\omega_1)_{2k+1} = 1$, $(\omega_1)_{2k+2} = 0$ for all $k \geqslant 0$, and $\omega_2 = T\omega_1$, and assign probability $\frac{1}{2}$ to each of these points. Let $g = 1$ or 0 depending upon whether an infinite number of the even coordinates has entry "0" or not, respectively. g is clearly a tail function and not an invariant function. Moreover, in this case g is a.s. equal to a function of X_1 and X_2 alone: $g = g_1(X_1, X_2)$ (P), and g_1 is $\mathfrak{T}$ measurable according to our conventions. However, if σ is the permutation interchanging 1 and 2, we obtain

$$g(\omega_1) = g_1(1, 0) = 1 \neq 0 = g_1(0, 1) = g(\omega_2) \quad (P).$$

It follows that g_1 is not $\mathfrak{S}$ measurable, although $\mathfrak{T}$ measurable. The trouble here is that $P(\{\sigma\omega_1\}) = 0$ but $P(\sigma^{-1}\{\sigma\omega_1\}) = \frac{1}{2}$.

Independent identically distributed variables satisfy the nonsingularity condition, and so does the more general class of exchangeable processes. The variables X_n are *exchangeable* if $P(\sigma^{-1}A) = P(A)$ for all $\sigma \in \Sigma$ and all events A. That an exchangeable process is strictly stationary is almost immediate: for finite dimensional cylinder sets exchangeability obviously implies the stationarity relation. Since $P(T^{-1}\cdot)$ is a measure and $P(T^{-1}A) = P(A)$ on cylinders, the equality holds for all sets generated by them, namely, all events. Thus (2) holds for exchangeable variables. The relation $\mathfrak{S} = \mathfrak{T}$ for exchangeable processes is mentioned in [4, p. 136]. The Hewitt-Savage theorem is a consequence of (2) via the well-known fact that the tail σ-field of an independent sequence is trivial.

2. **Proof of Theorem 1.** Because of strict stationarity, it may be assumed that the process is bilateral: $\{X_n, -\infty < n < \infty\}$, so that the shift T is a 1-1 measure preserving point transformation with a 1-1 measure preserving inverse T^{-1} [1, p. 456]. For f an $\mathfrak{S}$-indicator and g an $X_1, \ldots, X_{n-1}$ measurable function for fixed n, set

$$A = \{|f - g(X_1, \ldots, X_{n-1})| > \varepsilon\}.$$

Defining T_n (see §3), observe

$$T_n^{-1}A = \{|f - g(X_2, \ldots, X_n)| > \varepsilon\},$$

$$TT_n^{-1}A = \{|T^{-1}f - g(X_1, \ldots, X_{n-1})| > \varepsilon\}.$$

Choosing g with $P(A)$ small, $P(TT_n^{-1}A) = P_n(A)$ must be small, so $T^{-1}f$ can be approximated by the same functions approximating f. Thus $T^{-1}f = f$ or $f = Tf$, proving (1). To complete the proof, let f be $\mathfrak{T}$ measurable and let $\sigma \in \Sigma$. f differs from a strictly $\mathfrak{T}$ measurable function only on a P-null set, i.e., there is a function g such that $f = g$ a.e. (P) and for each integer $k \geqslant 1$ there is g_k depending only upon the coordinates $n \geqslant k$, and $g = g_k$ everywhere. Now since g does not depend on any finite initial string of coordinates $\sigma g = g$. Nonsingularity of σ easily shows $\sigma f = \sigma g$ a.s. (P) when $f = g$ a.s. (P). Therefore $\sigma f = \sigma g = g = f$ a.s. (P) and so f is $\mathfrak{S}$ measurable. The proof is complete.

3. **The invariant σ-field.** Let T_n be that element of Σ defined by: $(T_n\omega)_k = (\omega)_k$ for $k \geqslant n+1$; $(T_n\omega)_k = (\omega)_{k+1}$, $1 \leqslant k \leqslant n-1$; $(T_n\omega)_n = (\omega)_1$. Thus $T_n\omega$ and the shift $T\omega$ have the first $n-1$ coordinates identical. For any $\sigma \in \Sigma$, σP defines a measure by $(\sigma P)(A) = P(\sigma^{-1}A)$. Set $P_n = T_nP$ for each $n \geqslant 1$.

THEOREM 2. *A necessary and sufficient condition that $\mathfrak{S} \subset \mathfrak{I}$ (P) is that $P_n(C \cap V) \to P(C \cap V)$ as $n \to \infty$, for each cylinder set C and each $\mathfrak{S}$ set V.*

PROOF. Let C be any cylinder set determined by $X_1, X_2, \ldots, X_k$ for fixed $k \geqslant 1$, and let f be the indicator of an $\mathfrak{S}$-set. If $n \geqslant k+1$, $T_n^{-1}C = T^{-1}C$, and notice that Tf is also in $\mathfrak{S}$. A standard theorem about transformation of measures [3, p. 163] gives

$$\int_C f\,dP = \int_{T^{-1}C} Tf\,dP = \int_{T_n^{-1}C} Tf\,dP = \int_C Tf\,dP_n, \qquad n \geqslant k+1. \tag{4}$$

If the convergence condition above holds, the right side of (4) converges to $\int_C Tf\,dP$, and therefore the left side of (4) must equal this quantity. This equality is another way of saying

$$E(Tf|X_1, X_2, \ldots, X_k) = E(f|X_1, X_2, \ldots, X_k),$$

true for each positive integer k. As $k \to \infty$ the right and left terms of the preceding equality tend to Tf and f, respectively, by the martingale theorem since both functions are measurable on $X_1, X_2, \ldots$. Thus $Tf = f$ and f is invariant. Conversely, if f in $\mathfrak{S}$ is invariant, the left side of (4) can be written $\int_C Tf\,dP$, and then (4) says $P(C \cap V) = P_n(C \cap V)$, where f is the indicator of V, and n is large enough. This completes the proof of Theorem 2.

4. **Concluding remarks.** For processes with mutually independent and identically distributed variables or for exchangeable processes, $P_n(A) = P(A)$ for all events, and so Theorems 1 and 2 both apply. In the independent case it is well known that $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{I}$ are all trivial, but in the exchangeable case we obtain $\mathfrak{S} = \mathfrak{T} = \mathfrak{I}$ where the equality concerning $\mathfrak{I}$ may be new. The condition of Theorem 2 is a kind of continuity restriction for the transformations T_n^{-1} with respect to T^{-1}.

REFERENCES

1. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953. MR **15**, 445.

2. E. Hewitt and L. J. Savage, *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc. **80** (1955), 470–501. MR **17**, 863.

3. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N.J., 1950. MR **11**, 504.

4. J. Neveu, *Bases mathématiques du calcul des probabilités*, Masson, Paris, 1964. MR **33** #6659.

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