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A CHARACTERISATION OF RIESZ PROXIMITIES

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ABSTRACT. The purpose of this note is to characterise separated Riesz proximities generated by clusters.

1. **Introduction.** In the theory of proximity spaces of Efremovič [2], Smirnov [5] proved the following result.

A set X with a binary relation 'A close to B' written ($A\ \Pi\ B$) is a proximity space iff there exists a compact Hausdorff space Y in which X can be topologically embedded so that

$$A\ \Pi\ B \text{ in } X \text{ iff } \bar{A} \cap \bar{B} \neq \varnothing,$$

($\bar{A}$ denotes the closure of A).

The above result characterises Efremovič proximities. Lodato [3] characterised what are now known as Lodato proximities. The purpose of this note is to characterise Riesz proximities.

2. **Preliminaries.**

2.1. DEFINITIONS. Let X be a set, and c: $P(X) \to P(X)$ a map with the properties: $c(\varnothing) = \varnothing$, $A \subset c(A)$ for each A in $P(X)$ and $c(A \cup B) = c(A) \cup c(B)$ for A, B in $P(X)$. Then c is called a Čech closure operator and the pair (X, c) is called a Čech closure space. A closure space (X, c) is R_0 if for any two points x and y of X, $x \in c(y)$ implies $y \in c(x)$. It is called R_1 if for any x in X and A in $P(X)$, $c(x) \cap c(A) \neq \varnothing$ implies $x \in c(A)$.

Let (X, c) be a closure space and $Y \subset X$. Define c_Y: $P(Y) \to P(Y)$ by $c_Y(A) = c(A) \cap Y$ for $A \in P(Y)$. It is easy to verify that c_Y is a closure operator on Y. The pair (Y, c_Y) is called a subspace of (X, c). A mapping f of the closure space (Y_1, c_1) into the closure space (Y_2, c_2) is said to be *cl-continuous* if $f(c_1(A)) \subset c_2(f(A))\ \forall A \in P(Y_1)$. A one-one mapping f of the closure space (Y_1, c_1) onto the closure space (Y_2, c_2) is said to be a *cl-isomorphism* of (Y_1, c_1) onto (Y_2, c_2) if both f and f^{-1} are cl-continuous.

2.2. *Riesz proximity spaces*. As in Thron [4] we define a basic proximity space to be an abstract set X with a binary relation Π on its power set satisfying the following axioms: (i) $\Pi = \Pi^{-1}$, (ii) $A \cup B \in \Pi(C)$ iff $A \in \Pi(C)$ or $B \in \Pi(C)$, (iii) $A \cap B \neq \varnothing$ implies $A \in \Pi(B)$, (iv) $\varnothing \not\in \Pi(A)$ for every $A \in P(X)$.

Here $\Pi(A) = [B: (B, A) \in \Pi]$. When Π is a basic proximity on X, then the pair (X, Π) is called a basic proximity space. A proximity space (X, Π) is said

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to be separated if $x \in \Pi(y)$ implies $x = y$. A basic proximity Π on X is called Riesz proximity (RI-proximity) if it satisfies the following additional axiom:

$$\text{For } x \in X, A, B \in P(X), A, B \in \Pi(x) \text{ implies } A \in \Pi(B).$$

2.3. DEFINITION. The operator $c_\Pi(A) = [x: x \in \Pi(A)]$ is called the closure operator induced by the proximity Π.

2.4. THEOREM. *For every RI-proximity, c_Π is a Čech closure operator satisfying the R_1-axiom.*

PROOF. The fact that c_Π is a Čech closure operator is well known. Suppose $y \in c_\Pi(x) \cap c_\Pi(A)$. Then $y \in \Pi(x)$ and $y \in \Pi(A)$ which, in turn, implies $x \in \Pi(y)$ and $A \in \Pi(y)$. Since Π is a Riesz proximity, it follows that $x \in \Pi(A)$.

2.5. THEOREM. *Given any R_1-closure space (X, c), define Π_0 by $A\ \Pi_0\ B$ iff $c(A) \cap c(B) \neq \varnothing$. Then Π_0 is an RI-proximity relation on X and is compatible with the given closure, that is, $c_{\Pi_0} = c$.*

PROOF. That Π_0 is a basic proximity relation on $P(X)$ is a trivial consequence of the closure axioms. To prove that Π_0 is a Riesz proximity, suppose $A, B \in \Pi_0(x)$ where $x \in X$. Then $c(A) \cap c(x) \neq \varnothing$ and $c(B) \cap c(x) \neq \varnothing$. Since c is an R_1-closure, it follows that $x \in c(A) \cap c(B)$ and hence $A \in \Pi_0(B)$. Now

$$\begin{aligned} c_{\Pi_0}(A) &= [x \in X: x \in \Pi_0(A)] \\ &= [x \in X: c(x) \cap c(A) \neq \varnothing] = [x \in X: x \in c(A)] = c(A). \end{aligned}$$

The fact that c is an R_1-closure has been used to prove the above compatibility.

2.6. THEOREM. *Given an RI-proximity space (X, Π) and Π_0 defined by $A\ \Pi_0\ B$ iff $c_\Pi(A) \cap c_\Pi(B) \neq \varnothing$, we have that $A\ \Pi_0\ B$ implies $A\ \Pi\ B$ for all subsets A and B of X. Thus Π_0 is the smallest RI-proximity relation compatible with the closure in an R_1-closure space.*

PROOF. It follows immediately from 2.4 and 2.5.

Grills, clans and clusters. Grills were introduced by Choquet [1]. Below we give the definition of a grill. Elementary results on grills are mentioned in Thron [4].

2.7. DEFINITION. A family $\mathcal{G}$ of subsets of X satisfying the properties (i) $B \supset A \in \mathcal{G}$ implies $B \in \mathcal{G}$, (ii) $A \cup B \in \mathcal{G}$ implies $A \in \mathcal{G}$ or $B \in \mathcal{G}$, (iii) $\varnothing \notin \mathcal{G}$, is called a grill. For a fixed X, $\Gamma(X)$ will denote the set of all grills on X.

The following facts are evident: (i) For a proper grill $\mathcal{G}$ (nonempty), $A \subset X$ implies $A \in \mathcal{G}$ or $X \setminus A \in \mathcal{G}$. (ii) For a basic proximity space (X, Π), $\Pi(A)$ is a grill on X for all $A \in P(X)$.

2.8. DEFINITIONS. For a basic proximity (X, Π) a family $\mathcal{G}$ of subsets of X is called a Π-*clan* if it satisfies the following conditions: (i) $\mathcal{G}$ is a grill. (ii) $A, B \in \mathcal{G} \Rightarrow A \in \Pi(B)$.

A Π-clan $\mathcal{G}$ is said to be a maximal Π-clan if $\mathcal{G} \subset \mathcal{G}_1$, where $\mathcal{G}_1$ is another Π-clan, then $\mathcal{G} = \mathcal{G}_1$. A Π-clan $\mathcal{G}$ is called a Π-cluster if it satisfies the following additional condition: $\mathcal{G} \subset \Pi(A) \Rightarrow A \in \mathcal{G}$.

The following facts are immediate: (a) if $\mathcal{G}_1$ and $\mathcal{G}_2$ are clusters from X and $\mathcal{G}_1 \subset \mathcal{G}_2$, then $\mathcal{G}_1 = \mathcal{G}_2$. (b) If $A \cap B \neq \emptyset$ for every $B \in \mathcal{G}$, where $\mathcal{G}$ is a cluster, then $A \in \mathcal{G}$. (c) Every Π-cluster is a maximal Π-clan.

2.9. THEOREM. *A basic proximity space* (X, Π) *is an RI-proximity space iff* $\Pi(x)$ *is a cluster for all* $x \in X$.

PROOF. Suppose (X, Π) is a Riesz proximity space. For $x \in X$, surely $\Pi(x)$ is a grill. Let $A, B \in \Pi(x)$. Since Π is a Riesz proximity, it follows that $A \in \Pi(B)$. If $\Pi(x) \subset \Pi(A)$, then $x \in \Pi(A)$ and hence $A \in \Pi(x)$. The converse is an immediate consequence of the definition of the cluster.

2.10. COROLLARY. *If* $\mathcal{G}$ *is a cluster containing* x, *then* $\mathcal{G} = \Pi(x)$.

PROOF. The result follows from Definition 2.8 (a) and $\mathcal{G} \subset \Pi(x)$.

3. **Main result.** To state the main result we shall need the following

3.1. DEFINITION. A subset Y of a closure space (X, c) is regularly dense in X if given $F \subset X, p \notin c(F)$, there exists a subset E of Y with the property $p \in c(E) \subset X - c(F)$.

REMARK. If Y is regularly dense in X, then $c(Y) = X$.

3.2. THEOREM. *Let* X *be a set and* Π *a binary relation on* $P(X)$. *The following are equivalent*:

(I) *There exists an* R_1-*closure space* (Y, c) *and a mapping* f *of* X *into* Y *such that* $f(X)$ *is regularly dense in* Y, f *is a cl-isomorphism of* X *onto* $f(X)$ *satisfying* $c_{f(X)}(f(x)) = f(x)$ *and*

(*) $$A \, \Pi \, B \text{ in } X \text{ iff } c(f(A)) \cap c(f(B)) \neq \emptyset.$$

(II) Π *is a separated Riesz proximity satisfying the additional axiom*:

Given $A \, \Pi \, B$ *in* X, *there exists a cluster* $\mathcal{G}$ *to which both* A *and* B *belong*.

PROOF. Suppose (I) holds and define Π by (*). That Π is a basic proximity follows immediately from the properties of closure. Suppose $x \in \Pi(y)$. Then $c(f(x)) \cap c(f(y)) \neq \emptyset$. Since c is an R_1-closure, it follows that $f(x) \in c(f(y))$. Thus $f(x) \in c(f(y)) \cap f(X)$ that is, $f(x) \in c_{f(X)}(f(y)) = f(y)$. Since f is a cl-isomorphism of X onto $f(X)$, it follows that $x = y$. This proves that Π is a separated proximity. We next show that Π is a Riesz proximity. For $x \in X$, $A, B \in P(X)$, suppose $A, B \in \Pi(x)$. Then $c(f(x)) \cap c(f(A)) \neq \emptyset$ and $c(f(x)) \cap c(f(B)) \neq \emptyset$. That the closure operator is R_1 implies $f(x) \in c(f(A)) \cap c(f(B))$, that is, $A \in \Pi(B)$. It remains to prove for $(A, B) \in \Pi$ there exists a cluster to which both A and B belong. Now $(A, B) \in \Pi$, which implies that there exists a $y \in c(f(A)) \cap c(f(B))$. Define

$$\tau_y = \left[D \subset X : y \in c(f(D)) \right].$$

Surely A and B are in τ_y. We omit the details of the fact that τ_y is a cluster since they are quite similar to the ones given in Lodato [3].

For the converse suppose (II) holds. Given $x \in X$, the class $\Pi(x)$ is a cluster from X, by 2.9. For a subset A of X, let A^* be the set of all clusters to which A belongs. We will denote the set of all clusters from X by Y. Observe that

(3.2.1) $$(A \cup B)^* = A^* \cup B^*,$$

since clusters are grills.

Following Lodato [3], we say that a subset A of X absorbs a subset β of Y iff A belongs to every cluster in β, that is, $\beta \subset A^*$. For any subset β of Y, we define $c_1(\beta)$ by:

$\mathfrak{B} \in c_1(\beta)$ iff every subset E of X which absorbs β is in $\mathfrak{B}$.

It follows as in Lodato [3] that

(3.2.2) $$c_1(\beta_1 \cup \beta_2) = c_1(\beta_1) \cup c_1(\beta_2)$$

for all subsets β_1, β_2 in $P(Y)$ and $c_1(\mathfrak{B}) = \mathfrak{B}$ for every $\mathfrak{B}$ in Y.

Let f be the mapping which assigns to each x in X the cluster $\Pi(x)$ determined by it. This mapping is well defined. Define

(**) $$c(\beta) = \left(f^{-1}(\beta)\right)^* \cup c_1(\beta).$$

Observe that $c(f(A)) = A^*$. By definition

$$c(f(A)) = \left(f^{-1}(f(A))\right)^* \cup c_1(f(A)) = A^* \cup c_1(f(A)) = A^*,$$

since $c_1(f(A)) \subset A^*$. The inclusion $c_1(f(A)) \subset A^*$ is a consequence of the fact that A absorbs $f(A)$.

We now show that closure axioms are satisfied by the closure defined by (**).

Since $\beta \subset c_1(\beta)$, it follows that $\beta \subset c(\beta)$. The fact that $c(\varnothing) = \varnothing$ is trivial. (3.2.1), (3.2.2) and the fact that f^{-1} distributes on unions imply that $c(\beta_1 \cup \beta_2) = c(\beta_1) \cup c(\beta_2)$. Thus (Y, c) is a closure space. We shall next show that (Y, c) is an R_1-closure space. For $\mathfrak{B} \in Y, f^{-1}(\mathfrak{B})$ is either empty or equals x for some x in X. If $f^{-1}(\mathfrak{B}) = \varnothing$, then $c(\mathfrak{B}) = c_1(\mathfrak{B}) = \mathfrak{B}$. On the other hand, if $f^{-1}(\mathfrak{B}) = x$ for some x in X, then $\mathfrak{B} = \Pi(x)$. Hence

$$c(\mathfrak{B}) = \left(f^{-1}(\mathfrak{B})\right)^* \cup c_1(\mathfrak{B}) = \Pi(x) \cup \Pi(x) = \Pi(x) = \mathfrak{B}.$$

The separated character of Riesz proximity implies f is one-one. That f is a cl-isomorphism shall be accomplished by showing (i) $c_{f(X)}(f(A)) \supset f(c_\Pi(A))$ for every A in $P(X)$, and (ii) $f^{-1}(c_{f(X)}(f(A))) \subset c_\Pi(A)$ for each $A \subset X$. For (i), suppose $x \in c_\Pi(A)$. Then $A \in \Pi(x)$. Thus $\Pi(x) \in A^* = c(f(A))$ which, in turn, implies $\Pi(x) \in c_{f(X)}(f(A))$. In order to prove (ii), suppose $\mathfrak{B} \in c_{f(X)}(f(A))$. Then there exists an $x \in X$ such that $\mathfrak{B} = \Pi(x)$ and $\Pi(x) \in$

$c_{f(X)}(f(A)) = c(f(A)) \cap f(X)$. Thus $A \in \Pi(x)$, that is, $x \in c_\Pi(A)$.

$A \Pi B$ iff there exists a cluster to which both A and B belong, that is, $A^* \cap B^* \neq \varnothing$; thus $c(f(A)) \cap c(f(B)) \neq \varnothing$ iff $A \Pi B$.

It remains to check that $f(X)$ is regularly dense in Y. Suppose $\beta \subset Y$ and $\mathfrak{B}_0 \notin c(\beta) = (f^{-1}(\beta))^* \cup c_1(\beta)$. Then $f^{-1}(\beta) \notin \mathfrak{B}_0$ and there exists a subset A which absorbs β and does not belong to $\mathfrak{B}_0$. Since $\mathfrak{B}_0$ is, in particular, a grill, it follows that $A \cup f^{-1}(\beta) \notin \mathfrak{B}_0$. Using the fact that $\mathfrak{B}_0$ is a cluster, it follows that there exists a $B \in \mathfrak{B}_0$ such that $A \cup f^{-1}(\beta) \notin \Pi(B)$, that is, $A \notin \Pi(B)$ and $f^{-1}(\beta) \notin \Pi(B)$. Let $\mathfrak{B}$ be any element of B^*. Then $B \in \mathfrak{B}$ and hence $f^{-1}(\beta)$ and A do not belong to $\mathfrak{B}$. Thus it follows that $\mathfrak{B} \in Y \setminus c(\beta)$. Clearly $\mathfrak{B}_0 \in B^* = c(f(B)) \subset Y \setminus c(\beta)$. This completes the proof.

We end this section with an example of a Riesz proximity space in which a pair of proximal sets are contained in no cluster.

3.3. Example. Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$ and X_1 and X_2 are both infinite. Define a closure c on X by

$$c(D) = \begin{cases} D & \text{if } D \text{ is a finite subset of } X, \\ X_i \cup D & \text{if } D \text{ is infinite and } X_j \cap D \text{ is finite, } i, j = 1, 2, \text{ and } i \neq j, \\ X & \text{otherwise.} \end{cases}$$

(X, c) is an R_1-closure space. In fact, it is a T_1-topological space. We next define a binary relation Π on $P(X)$ as follows: $(D, E) \in \Pi$ iff $c(D) \cap c(E) \neq \varnothing$ or both D and E are infinite. (X, Π) is a Riesz proximity space. Moreover,

$$\Pi(x) = [\, D\colon x \in D \text{ or } X_i \cap D \text{ is infinite}\,]$$

if $x \in X_i$, $i = 1, 2$. That $\Pi(x)$ is a cluster follows from the fact that (X, Π) is a Riesz proximity space. Consider

$$\mathcal{G}^* = [\, D\colon D \text{ is an infinite subset of } X\,].$$

$\mathcal{G}^*$ is a maximal Π-clan. For $x_i \in X_i$, $i = 1, 2$, $\mathcal{G}^* \subset \Pi([x_1, x_2])$. However, $[x_1, x_2] \notin \mathcal{G}^*$. Thus $\mathcal{G}^*$ is not a cluster. Let $\mathfrak{B}$ be any Π-cluster. Then $\mathfrak{B} \not\subset \mathcal{G}^*$, for otherwise $\mathfrak{B} = \mathcal{G}^*$–a contradiction. Thus there exists an $x \in X$ such that $[x] \in \mathfrak{B}$ and this implies that $\mathfrak{B} = \Pi(x)$. Clearly $(X_1, X_2) \in \Pi$, but there exists no $\Pi(x)$ to which both X_1 and X_2 belong, for the existence of such an x would contradict the fact that $X_1 \cap X_2 = \varnothing$.

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