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THE DECIDABILITY OF THE THEORY OF BOOLEAN ALGEBRAS WITH THE QUANTIFIER "THERE EXIST INFINITELY MANY"

MARTIN WEESE

ABSTRACT. By using the decidability of the weak second order theory of linear order we get the decidability of the theory of Boolean algebras with the additional quantifier Q_0.

The quantifier Q_0 is defined as follows:
For any $\mathfrak{A}$,

$$\mathfrak{A} \models Q_0 x\varphi(x) \text{ iff } \operatorname{card}(\{a \in \mathfrak{A} | \mathfrak{A} \models \varphi(a)\}) \geqslant \aleph_0.$$

Ba denotes the elementary theory of Boolean algebras, formulated in the language with the following nonlogical symbols:

one ternary predicate $Uxyz$ (expressing the fact, that z is the sum of x and y),

one binary predicate Cxy (expressing the fact, that y is the complement of x) and

two constants o and e (denoting respectively zero and unit).

For the sake of simplicity we assume that $o \neq e$ is an axiom of Ba.

$\mathrm{Ba}(Q_0)$ denotes the theory of Boolean algebras in the language of Ba, with the additional quantifier Q_0.

LO denotes the elementary theory of linear order with least element, formulated in the language with the following nonlogical symbols:

one binary predicate $x < y$ (expressing the fact that x is less than y) and

one constant θ (denoting the least element).

LO^w denotes the weak second order theory of LO (that means, we add to the language the new variables $X, Y, Z, \ldots,$ ranging over finite sets, and the symbols $\in, \cup, \cap, \varnothing$, denoting respectively membership relation, union, intersection and empty set).

It is known that the weak second order theory of linear order is decidable (see [4] or [5]). Thus also LO^w is decidable.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. $\mathfrak{A} \equiv_0 \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ have the same theory in the language of Boolean algebras with the additional quantifier Q_0.

EXAMPLE. Let $\mathfrak{B}_\omega$ be the Boolean algebra of finite and cofinite subsets of ω.

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Then it is easily seen (for instance by using Ehrenfeucht games) that $\mathfrak{B}_\omega \equiv (\mathfrak{B}_\omega)^2$. We set

$$\varphi(x) =_{\mathrm{df}} Q_0 y U x y x \quad \text{and} \quad \psi =_{\mathrm{df}} \forall xy(Cxy \to \neg\varphi(x) \vee \neg\psi(y)).$$

$\varphi(x)$ expresses that there are infinitely many elements less than x and ψ expresses that for any element x, x or its complement cannot have infinitely many smaller elements. Then $\mathfrak{B}_\omega \models \psi$ and $(\mathfrak{B}_\omega)^2 \models \neg\psi$; thus we see that the theory $\mathrm{Ba}(Q_0)$ is more expressive than Ba.

The following theorem can be found in [2] or [1, §13, Theorem 5.1]:

THEOREM 1. *Let* $\mathfrak{A}$ *be any Boolean algebra. Then there is a Boolean algebra* $\mathfrak{B}$, *with* $\mathrm{card}(\{a|a \in \mathfrak{B}\}) \leqslant \aleph_0$, *such that* $\mathfrak{A} \equiv_0 \mathfrak{B}$.

Let $\mathfrak{M}$ be a linearly ordered set with first element. Then $\mathfrak{I}(\mathfrak{M})$ denotes the Boolean algebra, generated by the left-closed right-open intervals. One can show (see [3] or [6]):

THEOREM 2. *Let* $\mathfrak{A}$ *be a Boolean algebra with* $\mathrm{card}(\{a|a \in \mathfrak{A}\}) \leqslant \aleph_0$. *Then there is a linearly ordered set* $\mathfrak{M}$ *such that* $\mathfrak{A} \cong \mathfrak{I}(\mathfrak{M})$.

Let $\mathfrak{M}$ be a linearly ordered set with first element and $x \in \mathfrak{I}(\mathfrak{M})$. Then there are elements $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathfrak{M}$ (or $a_0, \ldots, a_n, b_0, \ldots, b_{n-1} \in \mathfrak{M}$) with $a_0 < b_0 < \cdots < a_n < b_n$ ($a_0 < b_0 < \cdots < a_n$) such that

$$x = \{y|a_0 \leqslant y < b_0\} \cup \cdots \cup \{y|a_n \leqslant y < b_n\}$$
$$(x = \{y|a_0 \leqslant y < b_0\} \cup \cdots \cup \{y|a_n \leqslant y\}).$$

Thus x can be coded by the two disjoint finite sets $X_0 = \{a_0, \ldots, a_n\}$ and $X_1 = \{b_0, \ldots, b_n\}$ ($X_0 = \{a_0, \ldots, a_n\}, X_1 = \{b_0, \ldots, b_{n-1}\}$).

$X_0 \cup X_1$ is the *support* of x (denoted by supp x). We set

$$\begin{aligned}\mathrm{Sup}(X, Y) =_{\mathrm{df}} X \cap Y = \varnothing \wedge \forall y(y \in Y \to \exists x(x \in X \wedge x < y))\\ \wedge \forall xy(x < y \wedge x \in X \wedge y \in X\\ \to \exists z(z \in Y \wedge x < z \wedge z < y))\\ \wedge \forall xy(x < y \wedge x \in Y \wedge y \in Y\\ \to \exists z(z \in X \wedge x < z \wedge z < y));\end{aligned}$$

that means, X and Y are the code of some element of the corresponding Boolean algebra. It is possible to describe union, complement, zero and unit with the help of codes.

Let φ be any formula of the language of $\mathrm{Ba}(Q_0)$. Then we have the following important fact:

$$\mathfrak{I}(\mathfrak{M}) \models Q_0 x\varphi(x)$$
$$\text{iff } \mathrm{card}(\bigcup\{\mathrm{supp}\, x|\mathfrak{I}(\mathfrak{M}) \models \varphi(x)\}) \geqslant \aleph_0.$$

Now we are in the position to define a function * from the set of formulas of $\mathrm{Ba}(Q_0)$ to the set of formulas of LO^w such that for every sentence φ of $\mathrm{Ba}(Q_0)$, $\mathrm{Ba}(Q_0) \vdash \varphi$ iff $\mathrm{LO}^w \vdash (\varphi)^*$.

$$(x = y)^* =_{\mathrm{df}} X_0 = Y_0 \wedge X_1 = Y_1;$$

$$(x = \mathrm{o})^* =_{\mathrm{df}} X_0 = \varnothing \wedge X_1 = \varnothing;$$

$$(x = e)^* =_{\mathrm{df}} X_0 = \{\theta\} \wedge X_1 = \varnothing;$$

$$\begin{aligned}(Uxyz)^* =_{\mathrm{df}} \forall x(x \in Z_0 \leftrightarrow \{[x \in X_0 \wedge \forall y(y \in Y_0 \wedge y < x \rightarrow &\\ \exists z(z \in Y_1 \wedge y < z \wedge z < x))] &\\ \vee [x \in Y_0 \wedge \forall y(y \in X_0 \wedge y < x \rightarrow &\\ \exists z(z \in X_1 \wedge y < z \wedge z < x))]\}) &\\ \wedge \forall x(x \in Z_1 \leftrightarrow \{[x \in X_1 \wedge \forall y(y \in Y_0 \wedge y < x \rightarrow &\\ \exists z(z \in Y_1 \wedge y < z \wedge z < x))] &\\ \vee [x \in Y_1 \wedge \forall y(y \in X_0 \wedge y < x \rightarrow &\\ \exists z(z \in X_1 \wedge y < z \wedge z < x))]\});\end{aligned}$$

$$\begin{aligned}(Cxy)^* =_{\mathrm{df}} \forall x(x \in X_0 \leftrightarrow x \in Y_1 \vee (x = \theta \wedge \theta \notin Y_0)) &\\ \wedge \forall x(x \in X_1 \leftrightarrow x \in Y_0 \wedge x \neq \theta);\end{aligned}$$

$$(\neg\varphi)^* =_{\mathrm{df}} \neg\varphi^*;$$

$$(\varphi \wedge \psi)^* =_{\mathrm{df}} \varphi^* \wedge \psi^*;$$

$$(\exists x\varphi(x)) =_{\mathrm{df}} \exists X_0 X_1 (\mathrm{Sup}(X_0, X_1) \wedge (\varphi(x))^*);$$

$$\begin{aligned}(Q_0 x\varphi(x)) =_{\mathrm{df}} \forall Y(\forall y(y \in Y \rightarrow \exists X_0 X_1 (\mathrm{Sup}(X_0, X_1) \wedge &\\ y \in X_0 \cup X_1 \wedge (\varphi(x))^*)) &\\ \rightarrow \exists Z(Y \neq Z \wedge \forall y(y \in Y \rightarrow y \in Z) \wedge &\\ \forall y(y \in Z \rightarrow \exists X_0 X_1 (\mathrm{Sup}(X_0, X_1) \wedge y \in X_0 \cup X_1 \wedge (\varphi(x))^*))).\end{aligned}$$

Let φ be any sentence of the language of $\mathrm{Ba}(Q_0)$. It follows immediately from the construction, that: if $\mathrm{Ba}(Q_0) \vdash \varphi$, then $\mathrm{LO}^w \vdash (\varphi)^*$. Together with Theorem 1 and Theorem 2 we also get: if $\mathrm{LO}^w \vdash (\varphi)^*$, then $\mathrm{Ba}(Q_0) \vdash \varphi$. Now it follows from the decidability of LO^w, that also $\mathrm{Ba}(Q_0)$ is decidable.

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SEKTION MATHEMATIK, BEREICH LOGIK, HUMBOLDT-UNIVERSITAT, 1086 BERLIN, POSTFACH 1297, DDR