ON CERTAIN WEIGHTED PARTITIONS AND FINITE SEMISIMPLE RINGS

L. B. RICHMOND AND M. V. SUBBARAO

Abstract. Let \( k \) be a fixed integer \( \geq 1 \) and define \( \tau_k(n) = \sum d^k/n \). Thus \( \tau_1(n) \) is the ordinary divisor function and \( \tau_k(n) \) is the number of \( k \)th powers dividing \( n \). We derive the asymptotic behaviour as \( n \to \infty \) of \( P_k(n) \) defined by

\[
\sum_{n=0}^{\infty} P_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\tau_k(n)}.
\]

Thus \( P_k(n) \) is the number of partitions of \( n \) where we recognize \( \tau_k(m) \) different colours of the integer \( m \) when it occurs as a summand in a partition. The case \( k = 2 \) is of special interest since the number \( f(n) \) of semisimple rings with \( n \) elements when \( n = q_1^1q_2^2 \ldots \) is given by \( f(n) = P_2(l_1)P_2(l_2) \ldots \).

1. Let \( k \) be a fixed integer \( \geq 1 \) and define

\[
\tau_k(n) = \sum_{d^k/n} 1.
\]

Thus \( \tau_1(n) \) is the ordinary divisor function and \( \tau_k(n) \) is the number of \( k \)th powers dividing \( n \). We shall derive the asymptotic behaviour of \( p_k(n) \) defined by

\[
\sum_{n=0}^{\infty} p_k(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-\tau_k(n)}.
\]

Thus \( p_k(n) \) is the number of partitions of \( n \) where we recognize \( \tau_k(m) \) different colours of the integer \( m \) when it occurs as a summand in a partition. The case \( k = 2 \) is of special interest since the number of semisimple rings with \( n \) elements \( f_2(n) \), when

\[
n = p_1^p p_2^{p^2} \ldots
\]

is given by \( f_2(n) = p_2(p_1)p_2(p_2) \ldots \) [1]. Also, when \( k \) is large, we expect \( p_k(n) \) to approach \( p(n) \), the number of ordinary partitions.

A generating function for \( \tau_k(n) \) is given by

\[
\sum_{n=1}^{\infty} \tau_k(n)n^{-s} = \zeta(s)\zeta(ks).
\]

Lemma 1. If \( k > 1 \),

\[
\sum_{n=1}^{N} \tau_k(n) = \zeta(k)N + O \left\{ N^{1/k} \right\}
\]

Received by the editors October 24, 1974.

AMS (MOS) subject classifications (1970). Primary 10J20; Secondary 15A17, 10A45.

© American Mathematical Society 1977
and
\[
\sum_{n=1}^{N} \tau_1(n) = N \log N + (2\gamma - 1)N + O\{N^{1/2}\}
\]

where \(\gamma\) is Euler’s constant.

**Proof.** The case \(k = 1\) is classical; see, for example, Theorem 320 in [2, p. 264]. The other cases are similar:
\[
\sum_{n<x} \tau_k(n) = \sum_{d^k<x} \left\lfloor \frac{x}{d^k} \right\rfloor = x \sum_{d^k<x} 1 + O\left\{x^{1/k}\right\} = x\psi(k) + O\left\{x^{1/k}\right\}.
\]

Lemma 1 shows that if we let \(F_\tau(x) = \sum_{n<x} \tau_k(n)\), then \(F_\tau(2x) = O\{F_\tau(x)\}\) as \(x \to \infty\).

Let us define the function \(f_\tau\) for real \(x > 0\) by
\[
f_\tau(x) = \sum_{n=1}^{\infty} \tau_k(n)e^{-xn}.
\]

We define \(\alpha\) throughout this paper to be the unique solution of
\[
n = \sum_{m=1}^{\infty} \tau_k(m)m(e^{\alpha m} - 1)^{-1}.
\]

**Theorem 1.** Let \(m\) be any fixed integer \(\geq 3\). Let \(k \geq 1\) be a fixed integer. Then
\[
p_k(n) = (2\pi B_2)^{-1/2} \exp\left\{\alpha n - \sum_{m=1}^{\infty} \tau_k(n)\log(1 - e^{-\alpha n})\right\}
\]
\[
\times \left[1 + \sum_{\rho=1}^{m-2} D_\rho + O\left\{f_\tau^{1-2m/3}(\alpha)\right\}\right].
\]

Here we define \(B_\mu = B_\mu(n)\) (\(\mu = 2, 3, \ldots\)) by
\[
B_\mu = \sum_{m=1}^{\infty} \tau_k(m)m^{\mu}g_\mu(e^{\alpha m})(e^{\alpha m} - 1)^{-\mu}
\]
where \(g_\mu(x)\) is a certain polynomial (the same as in [3] or the \(g_\mu^*\) of Roth and Szekeres [4]) of degree \(\mu - 1\) and, in particular, \(g_1(x) = 1\) and \(g_2(x) = x\) so that
\[
B_2 = \sum_{m=1}^{\infty} \tau_k(m)m^2e^{\alpha m}(e^{\alpha m} - 1)^{-2}.
\]

Finally \(D_\rho\) (\(\rho = 1, 2, \ldots\)) is defined by
\[
D_\rho = B_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_{2\rho}=2}^{\infty} d_{\mu_1} \cdots d_{\mu_{2\rho}} B_{\mu_1} B_{\mu_2} \cdots B_{\mu_{2\rho}},
\]
the summation being subject to \(\mu_1 + \mu_2 + \cdots + \mu_{2\rho} = 12\rho\), and where the \(d_\mu\)'s are certain numerical constants.

**Proof.** It is only necessary to note that the conditions of Theorem 1.1 of [3] hold. For convenience we restate the theorem here in terms of the notation of ...
the present paper. We say that $\tau_k$ is a $P$-function if the integers $l$ such that $\tau_k(l) \neq 0$ do not have a common factor $> 1$ for all sufficiently large $l$. Then Theorem 1.1 of [3] says:

Let $\tau_k(n)$ have properties (I) and (II). Suppose that $\tau_k(n)$ is a $P$-function and that $\min_{\tau_k(l) \neq 0} \tau_k(l) > 0$. Suppose furthermore that

$$\lim_{x \to \infty} \frac{\log F_x(x)}{\log \log x} > 0.$$ 

Let $m$ be any fixed integer $> 2$. Then

$$P_k(n) = (2\pi B_2)^{1/2} \exp \left\{ an - \sum_{l=1}^{\infty} \tau_k(l) \log(1 - e^{al}) \right\}$$

$$\times \left[ 1 + \sum_{l=1}^{m-2} D_\varepsilon + O \left\{ f_r^{-2m/3}(a) \right\} \right].$$

It is not necessary to define conditions (I) and (II) since it is shown in [3] that they hold when $F_x(2x) = \mathcal{O}(F_x(x))$ holds, which we have seen does hold. It is clear that $\tau_k$ is a $P$-function and, furthermore, $\tau_k(l) > 1$. Also the last condition of Theorem 1.1 holds by Lemma 1. Theorem 1 now follows immediately.

2. In this section we determine the asymptotic behaviour of $p_k(n)$ in terms of elementary functions. First of all, from the Mellin inversion formula,

$$n = \sum_{m=1}^{\infty} \tau_k(m)m(e^{am} - 1)^{-1} = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \tau_k(m)me^{-aml}$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t} \Gamma(t)\xi(t) \sum_{m=1}^{\infty} \tau_k(m)m^{1-t} \, dt$$

for $\sigma > 2, |\arg \alpha| < \pi/2$.

It is well known that (equation (1.2))

$$\sum_{m=1}^{\infty} \tau_k(m)m^{-t} = \xi(t)\xi(ik);$$

hence,

$$n = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t} \Gamma(t)\xi(t)\xi(t-1)\xi((t-1)k) \, dt. \quad (2.1)$$

**Lemma 2.1.** Let $\alpha$ be defined by equation (1.3) with $k = 1$. Then

$$\alpha = \frac{\pi}{\sqrt{12}} n^{-1/2} \log^{1/2} n \left[ 1 + O \left( \frac{\log \log n}{\log n} \right) \right].$$

Let $k \geq 2$. Then with

$$b_k = \frac{\Gamma(1 + 1/k)\xi(1 + 1/k)\xi(1/k)}{\left( \xi(2)\xi(k) \right)^{1/2 + 1/2k} 2k},$$
\[ \alpha = n^{-1/2}(\xi(2)\xi(k))^{1/2} + n^{1/2k-1}\sqrt{\xi(2)\xi(k)} b_k + n^{-1}/8 + O\left(n^{-1/2k-1}\right). \]

**Proof.** The singularities of \( \alpha^{-1}\Gamma(t)\xi(t)\xi(t-1)\xi((t-1)k) \) for \( k = 1, 2, \ldots \) with real part of \( t > 0 \) are at \( t = 0, 1, 2 \) and \( 1 + 1/k \). For \( k = 1 \) there is a double pole, hence the residue at 2 must be evaluated as

\[ (2.2) \quad \left[ \left( \frac{d}{dt}\right) \left( \frac{\alpha^{-1}\Gamma(t)\xi(t)}{(t-2)} \right) + 2\alpha^{-1}\xi(t)\Gamma(t)(\xi(t-1) - 1/(t-2)) \right]_{t=2}. \]

Let us consider the case \( k = 1 \) first. From equation (2.1) and equation (2.2) and the relations

\[ \Gamma'(1) = -\gamma, \quad \left[ \xi(s) - 1/(s-1) \right]_{s=1} = \gamma, \quad \xi(2) = \pi^2/6, \]

we obtain that

\[ n = \frac{\pi^2}{6} \frac{\log(1/\alpha)}{\alpha^2} + O\left(\alpha^{-2}\right). \]

The first part of the lemma follows from this.

For \( k = 2, 3, \ldots \) we obtain from equation (2.1) that

\[ n = \alpha^{-2}\xi(2)\xi(k) + \alpha^{-1-1/k}\Gamma\left(1 + \frac{1}{k}\right)\xi\left(1 + \frac{1}{k}\right)\xi\left(\frac{1}{k}\right)/k \]

\[ + \alpha^{-1}\xi(0) + O\left(1\right) \]

and the second part of the lemma follows routinely from this using the fact that \( \xi(0) = -1/2 \).

**Lemma 2.2.** Let \( k = 1 \). Then

\[ \sum_{m=1}^{\infty} \tau_k(m) \log(1 - e^{-am}) = \frac{\pi^2}{6\alpha} \log \frac{1}{\alpha} + O\left(\alpha^{-1}\right). \]

Let \( k = 2, \ldots \); then

\[ - \sum_{m=1}^{\infty} \tau_k(m) \log(1 - e^{-am}) \]

\[ = \alpha^{-1}\xi(k)\xi(2) + \frac{\xi(1/k)}{k} \xi\left(1 + \frac{1}{k}\right)\Gamma\left(\frac{1}{k}\right)\alpha^{-1/k} \]

\[ + \frac{1}{4} \log \frac{1}{\alpha} - \frac{(1+k)}{2} \xi'(0) + O\left(\alpha\right). \]

**Proof.** We derive as above that

\[ - \sum_{m=1}^{\infty} \tau_k(m) \log(1 - e^{-am}) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \alpha^{-i}\Gamma(t)\xi(1+t)\xi(t)\xi(tk) \, dt, \]

and then proceed as in the proof of Lemma 2.1. (Note \( \xi'(0) = -\frac{1}{2}\log 2\pi \) and \( \Gamma(t) = 1/t - \gamma + \ldots \).)

**Lemma 2.3.** Let \( k = 1 \). Then

\[ B_2 = 2\xi(2)\alpha^{-3} \log(1/\alpha) + O\left(\alpha^{-3}\right). \]

Let \( k = 2, 3, \ldots \). Then
$B_2 = 2\alpha^{-3\xi(2)}\xi(k) + O\{\alpha^{-2-1/k}\}$.

**Proof.** Note that

$$\sum_{m=1}^{\infty} \tau_k(m)m^2e^{\alpha m}(e^{\alpha m} - 1)^{-2} = - \frac{d}{d\alpha} \sum_{m=1}^{\infty} \tau_k(m)(e^{\alpha m} - 1)^{-1}$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t-1}\Gamma(t)\xi(t)\xi(t-1)\xi((t-1)k) \, dt$$

and the proof proceeds as in Lemmas 2.1 and 2.2.

From Lemmas 2.1, 2.2, 2.3 and Theorem 1, we now obtain, using the facts that $\xi'(0) = -\frac{1}{2}\log 2\pi$ and $\xi(2) = \pi^2/6$.

**Theorem 2.1.** As $n \to \infty$,

$$\log p_1(n) = \frac{\pi}{\sqrt{3}} n^{1/2}\log^{1/2}n \left[ 1 + O\left\{ \frac{(\log \log n)^2}{\log n} \right\} \right].$$

Let $k = 2, 3, \ldots$. Then as $n \to \infty$,

$$p_k(n) = \exp \left[ \frac{1}{2\pi n^{1/2}} \left( \frac{\xi(k)}{6} \right)^{1/2} + \frac{\Gamma(1 + 1/k)\xi(1 + 1/k)\xi(1/k)}{(\xi(2)\xi(k))^{1/2}} n^{1/2k} \right.$$ 

$$- \frac{n^{1/k-1/2}}{4k^2} \frac{\Gamma^2(1 + 1/k)\xi^2(1 + 1/k)\xi^2(1/k)}{(\xi(k)\xi(2))^{1/2+1/k}} + \left( \frac{1 + k}{4} \right)\log 2\pi \right]$$

$$\times \left[ 1 + O\left\{ n^{-1/2k} \right\} \right].$$

Note one could obtain as many terms in the asymptotic expansion as required. However, we have not discovered a general formula.

**Corollary** Let $f_2(n)$ denote the number of semisimple rings with $n = p^m$ elements. Then with

$$A = \exp \left( - \frac{9}{4\pi^4} \Gamma^2(1.5)\xi^2(1.5)\xi^2(.5) \right) \pi^{3/5} 12^{-1/4} \log^{5/8} p,$$

$$f_2(n) \sim A \log^{-5/8} n \exp \left( \frac{\pi^2}{3} \left( \frac{\log n}{\log p} \right)^{1/2} \right.$$ 

$$+ \frac{6^{1/2}}{\pi} \Gamma(1.5)\xi(1.5)\xi(.5) \left( \frac{\log n}{\log p} \right)^{1/4} \right).$$

**Proof.** It is only necessary to note that if $n = p^m$ then $f_2(n) = p^2(m)$ (see e.g. (1.1)).
This corollary provides the asymptotic formula suggested by Knopfmacher on p. 23 of [5]. In [5] it is also shown that \( p_2(n) \) is the number of nonisomorphic semisimple \( n \)-dimensional algebras over the Galois field \( \text{GF}(p^n) \), \( p \) a prime.

This corollary shows that the behaviour of \( f_2(n) \) is very irregular, since if \( n = p \), a prime, then \( f_2(n) = 1 \). The average behaviour of \( f_2(n) \) was originally discussed by Connell [1]. Recently Knopfmacher [5] showed that

\[
\sum_{n \leq x} f_2(n) = \alpha_1 x + \alpha_2 x^{1/2} + O \left( x^{1/3} \log^2 x \right)
\]

where

\[
\alpha_1 = \prod_{rm^2 > 1} \zeta \left( \frac{1}{2} \right) \prod_{rm^2 > 1} \zeta \left( \frac{1}{2} \right)
\]
\[
\alpha_2 = \left(\frac{1}{2}\right) \prod_{rm^2 > 1} \zeta \left( \frac{1}{2} \right).\]

However, Knopfmacher [6, Theorem E] has shown that for any \( \epsilon > 0 \) there is an integer \( n_0(\epsilon) \) such that

\[
f_2(n) < 6^{\frac{1}{2}} \left( 1 + \epsilon \right) \frac{\log n}{\log \log n} \quad \text{for all} \quad n > n_0(\epsilon),
\]

while

\[
f_2(n) > 6^{\frac{1}{2}} \left( 1 - \epsilon \right) \frac{\log n}{\log \log n} \quad \text{for infinitely many} \quad n.
\]

Moreover,

\[
f_2(n) < 6^{\frac{1}{2}} \left( 1 + \epsilon \right) \frac{\log \log n}{\log n} \quad \text{for "almost all"} \quad n,
\]

i.e. for all \( n \) outside some set of asymptotic density zero.

Since any partition of \( n \) when a one is added to it gives a partition of \( n + 1 \), it is clear that \( p_k(n) \) is monotonic increasing. Furthermore, one may derive from Theorem 1, in a manner similar to that of Roth and Szekeres [4], that if \( p_k^{(0)}(n) \) denotes the \( l \)th difference of \( p_k(n) \) that \( p_k^{(0)}(n) \sim \alpha p_k(n) \); hence all the differences of \( p_k(n) \) are positive for \( n \) sufficiently large. Below we give a short table of values of \( p_2(n) \) which are useful for computing \( f_2(n) \) and the comparison between the asymptotic and true value for certain values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p_2(n) )</th>
<th>( n )</th>
<th>( p_2(n) )</th>
<th>( n )</th>
<th>( p_2(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>11</td>
<td>79</td>
<td>21</td>
<td>1549</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>12</td>
<td>115</td>
<td>22</td>
<td>2025</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>13</td>
<td>154</td>
<td>23</td>
<td>2600</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>14</td>
<td>213</td>
<td>24</td>
<td>3377</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>15</td>
<td>284</td>
<td>25</td>
<td>4306</td>
</tr>
<tr>
<td>6</td>
<td>13</td>
<td>16</td>
<td>391</td>
<td>26</td>
<td>5523</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>17</td>
<td>514</td>
<td>27</td>
<td>7000</td>
</tr>
<tr>
<td>8</td>
<td>29</td>
<td>18</td>
<td>690</td>
<td>28</td>
<td>8922</td>
</tr>
<tr>
<td>9</td>
<td>40</td>
<td>19</td>
<td>900</td>
<td>29</td>
<td>11235</td>
</tr>
<tr>
<td>10</td>
<td>58</td>
<td>20</td>
<td>1197</td>
<td>30</td>
<td>14196</td>
</tr>
<tr>
<td>$n$</td>
<td>True Value of $p_2(n)$</td>
<td>Asymptotic Value of $p_2(n)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-----------------------------</td>
<td>-----------------------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>231 412 7129</td>
<td>$2.55495 \times 10^9$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>261 229 585 686401</td>
<td>$2.83594 \times 10^{14}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>246 910 805 791 4492823</td>
<td>$2.65888 \times 10^{18}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>616 439 413 088 071 894 2607</td>
<td>$6.60456 \times 10^{21}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>645 864 386 271 246 677 988 3980</td>
<td>$6.89497 \times 10^{24}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We set $\zeta(1/2) = -1.460$, $\zeta(1.5) = 2.612$ and $\Gamma(1.5) = .8862$ in the asymptotic expression. Since the relative error is $O\left(n^{1/4}\right)$ we cannot expect a rapid decrease in the relative error. The exact values were computed using the recurrence

$$np_2(n) = \sum_{k=1}^{n} a(k)p_2(n - k) \quad \text{where} \quad a(k) = \sum_{d|k} d r_k(d).$$

This recurrence is obtained by taking the logarithmic derivative of equation (1.0) and comparing coefficients.

REFERENCES