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COMPARISON OF EIGENVALUES FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with comparison theorems of an "integral type" concerning the smallest positive eigenvalues of systems of linear differential equations.

1. **Introduction**. In [1], Travis derived comparison theorems of an "integral type" concerning the smallest positive eigenvalues, λ_0 and Λ_0, respectively, of the following differential equations:

$$(1.1)\qquad \begin{cases} \left[a(x)u^{(n)}\right]^{(n)} + (-1)^{n+1}\lambda b(x)u = 0, \\ u^{(i)}(\alpha) = \left[au^{(n)}\right]^{(i)}(\beta) = 0, \qquad i = 0,1,\ldots,n-1, \end{cases}$$

$$(1.2)\qquad \begin{cases} \left[c(x)v^{(n)}\right]^{(n)} + (-1)^{n+1}\Lambda d(x)u = 0, \\ v^{(i)}(\alpha) = \left[cv^{(n)}\right]^{(i)}(\beta) = 0, \qquad i = 0,1,\ldots,n-1, \end{cases}$$

where $a(x)$, $c(x)$ are positive functions of class $C^{(n)}[\alpha, \beta]$, $b(x)$ and $d(x)$ are of class $C[\alpha, \beta]$. The purpose of this paper is to show that such theorems can also be extended to certain systems of linear differential equations.

In particular, we shall be concerned mainly with systems of the following form:

$$(1.3)\qquad \begin{cases} \left[A(x)\mathbf{u}^{(n)}\right]^{(n)} + (-1)^{n+1}\lambda B(x)\mathbf{u} = \mathbf{0}, \\ \mathbf{u}^{(i)}(\alpha) = \left[A\mathbf{u}^{(n)}\right]^{(i)}(\beta) = \mathbf{0}, \qquad i = 0,1,\ldots,n-1, \end{cases}$$

where $\mathbf{u}(x)$ is an $m \times 1$ column vector function, $A(x)$ is an $m \times m$ diagonal matrix whose elements are positive functions of class $C^{(n)}[\alpha, \beta]$ and $B(x)$ is an $m \times m$ matrix whose elements are functions of class $C[\alpha, \beta]$. In the final section of this paper, we shall also mention some other systems which allow similar developments as those for (1.3).

In what follows, matrix notations will be used throughout. Moreover, all *operations* and *inequalities* related to matrices are *elementwise* defined; in particular, $M(x) > 0$ means all its elements are positive and $\int_\alpha^\beta M(x)\,dx$

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denotes the matrix of integrals of respective elements of $M(x)$. $\mathfrak{B}$ will be used to denote the real Banach space $\{u \in C^{2n}[\alpha, \beta] | u^{(i)}(\alpha) = 0, i = 0, 1, \ldots, n-1\}$ with norm $\|u\| = \max_{x \in [\alpha, \beta]}\{|u^{(i)}(x)|, i = 0,1, \ldots, 2n\}$ while $\mathfrak{P}_1$ and $\mathfrak{P}_2$ will denote, respectively, the subsets $\{u \in \mathfrak{B} | u(x) \geqslant 0$ on $[\alpha, \beta]\}$ and $\{u \in \mathfrak{B} | u'(x) \geqslant 0$ on $[\alpha, \beta]\}$ of $\mathfrak{B}$. It is then easily verified that the cartesian products $\mathfrak{P}_1^m$ and $\mathfrak{P}_2^m$ are positive cones [2] of the product Banach space $\mathfrak{B}^m$, where m is a positive integer. Finally, we mention that the theory of u_0-*positive* operators and the abstract comparison theorem stated in [1] or [3] will be used freely in the following sections.

2. **Green's functions.** Let $a(x)$, $c(x)$ be positive functions of class $C^{(n)}[\alpha, \beta]$. Let $G_a(x, s)$, $G_c(x, s)$ be, respectively, the Green's functions of the following selfadjoint operators:

$$\begin{cases} L_a[u] = (-1)^n\left[a(x)u^{(n)}\right]^{(n)}, \\ u^{(i)}(\alpha) = \left[au^{(n)}\right]^{(i)}(\beta) = 0, \qquad i = 0,1, \ldots, n-1, \end{cases} \tag{2.1}$$

$$\begin{cases} L_c[v] = (-1)^n\left[c(x)v^{(n)}\right]^{(n)}, \\ v^{(i)}(\alpha) = \left[cv^{(n)}\right]^{(i)}(\beta) = 0, \qquad i = 0,1, \ldots, n-1. \end{cases} \tag{2.2}$$

The Green's functions exist, are nonnegative [1] and satisfy some "positivity" properties. We list below those properties that concern us here and the proofs can be found in [1].

LEMMA 2.1. *If* $\int_\alpha^x a^{-1}(s)\,ds \leqslant \int_\alpha^x c^{-1}(s)\,ds$ *on* $[\alpha, \beta]$, *then* $0 \leqslant G_a(x, s) \leqslant G_c(x, s)$ *on* $[\alpha, \beta] \times [\alpha, \beta]$.

LEMMA 2.2. *Suppose* $b(x) \in C[\alpha, \beta]$ *and is positive, then for every closed interval* $I \subset [\alpha, \beta]$, *a positive number* δ *exists such that*

$$\delta \int_I G_a(x, s)b(s)\,ds \leqslant \int_\alpha^\beta G_a(x, s)b(s)\,ds \quad \textit{for } x \in [\alpha, \beta].$$

LEMMA 2.3. *If* $\int_\alpha^x a^{-1}(s)\,ds \leqslant \int_\alpha^x c^{-1}(s)\,ds$ *and* $b(x) \geqslant 0$ *on* $[\alpha, \beta]$, *then for any nonzero* $u \in \mathfrak{P}_1$,

$$\int_\alpha^\beta G_a(x, s)b(s)u(s)\,ds \leqslant \int_\alpha^\beta G_c(x, s)b(s)u(s)\,ds \quad \textit{for } x \in [\alpha, \beta].$$

LEMMA 2.4. *If* $b(x), d(x) \in C[\alpha, \beta]$ *and* $\int_x^\beta b(s)\,ds \leqslant \int_x^\beta d(s)\,ds$ *on* $[\alpha, \beta]$, *then for any nonzero* $u \in \mathfrak{P}_2$,

$$\int_\alpha^\beta G_a(x, s)b(s)u(s)\,ds \leqslant \int_\alpha^\beta G_a(x, s)d(s)u(s)\,ds \quad \textit{for } x \in [\alpha, \beta].$$

LEMMA 2.5. *If* $0 < c(x) \leqslant a(x)$ *and* $\int_x^\beta b(s)\,ds \leqslant \int_x^\beta d(s)\,ds$ *on* $[\alpha, \beta]$, *then for any nonzero* $u \in \mathcal{P}_2$,

$$\int_\alpha^\beta G_a(x, s)b(s)u(s)\,ds \leqslant \int_\alpha^\beta G_c(x, s)d(s)u(s)\,ds \quad \text{for } x \in [\alpha, \beta].$$

3. **Existence of smallest eigenvalues.** Consider system (1.3). Let a_i, $1 \leqslant i \leqslant m$, be the diagonal elements of $A(x)$ and let $G_{a_i}(x, s)$ be its corresponding Green's functions as defined in §2. Let $G_A(x, s)$ be the diagonal matrix whose diagonal elements are G_{a_i}, $1 \leqslant i \leqslant m$, and consider the compact operator T_{AB} on the product Banach space $\mathcal{B}^m$ defined by

$$T_{AB}\mathbf{u}(x) = \int_\alpha^\beta G_A(x, s)B(s)\mathbf{u}(s)\,ds.$$

THEOREM 3.1. *If* $B(x) = [b_{ij}(x)] > 0$, *then* T_{AB} *is* u_0-*positive with respect to* $\mathcal{P}_1^m$.

PROOF. We will show that for any nonzero $u \in \mathcal{P}_1^m$, real $k, l > 0$ exist such that

$$k\mathbf{u}_0 \leqslant T_{AB}\mathbf{u} \leqslant l\mathbf{u}_0 \tag{3.1}$$

where $\mathbf{u}_0(x) = \int_\alpha^\beta G_A(x, s)J_m\,ds$, $J_m = \operatorname{col}(1, 1, \ldots, 1)_{1\times m}$.

Note first that if nonzero $\mathbf{u}$ belongs to $\mathcal{P}_1^m$, then each component $(B\mathbf{u})_i$ of $B\mathbf{u}$ is nonzero and nonnegative. Hence a closed subinterval $I_i \subset [\alpha, \beta]$ can be found such that $(B\mathbf{u})_i(x) \geqslant k_i > 0$ for $x \in I_i$. Consequently, by Lemma 2.2, a positive constant δ_i exists so that

$$\begin{aligned} k_i\delta_i\int_{I_i} G_{a_i}(x, s)\,ds &\leqslant k_i\int_\alpha^\beta G_{a_i}(x, s)\,ds \leqslant \int_\alpha^\beta G_{a_i}(x, s)(B\mathbf{u})_i(s)\,ds \\ &\leqslant \max_{\alpha \leqslant t \leqslant \beta} |(B\mathbf{u})_i(t)| \int_\alpha^\beta G_{a_i}(x, s)\,ds \equiv l_i\int_\alpha^\beta G_{a_i}(x, s)\,ds. \end{aligned}$$

Now let $k = \min\{k_i | 1 \leqslant i \leqslant m\}$ and $l = \max\{l_i | 1 \leqslant i \leqslant m\}$. Then $k\mathbf{u}_0 \leqslant T_{AB}\mathbf{u} \leqslant l\mathbf{u}_0$ as required. Q.E.D.

THEOREM 3.2. *If each element of* B *is not identically zero and if* $\int_\alpha^\beta B(s)\,ds \geqslant 0$ *on* $[\alpha, \beta]$, *then* T_{AB} *is* u_0-*positive with respect to* $\mathcal{P}_2^m$.

PROOF. Note that if nonzero $\mathbf{u}$ belongs to $\mathcal{P}_2^m$, then a nonempty subset M of $\{1, 2, \ldots, m\}$ can be found such that $\mathbf{u}_j \not\equiv 0$ for each $j \in M$ and $\mathbf{u}_j \equiv 0$ for each $j \notin M$. An argument similar to that given by Travis [1, p. 369] will establish that for each nonzero $\mathbf{u} \in \mathcal{P}_2^m$, real k_i, $l_i > 0$ $(1 \leqslant i \leqslant m)$ exist such that

$$\begin{aligned} k_i\int_\alpha^\beta G_{a_i}(x, s)\sum_{j=1}^m b_{ij}(s)\,ds &\leqslant \int_\alpha^\beta G_{a_i}(x, s)\sum_{j=1}^m b_{ij}(s)\mathbf{u}_j(s)\,ds \\ &\leqslant l_i\int_\alpha^\beta G_{a_i}(x, s)\sum_{j=1}^m b_{ij}(s)\,ds. \end{aligned}$$

The proof is then complete by choosing $k = \min\{k_i | 1 \leqslant i \leqslant m\}$, $l = \max\{l_i | 1 \leqslant i \leqslant m\}$ and $\mathbf{u}_0(x) = \int_\alpha^\beta G_A(s, s)B(s)J_m\, ds$, $J_m = \text{col}(1, 1, \ldots, 1)$. Q.E.D.

According to a theorem [1, 2] for u_0-positive operators, if $B(x) > 0$ ($\int_x^\beta B(s)\, ds \geqslant 0$ and each component $b_{ij}(x)$ of $B(x)$ is not identically zero) on $[\alpha, \beta]$, then T_{AB} has exactly one (normalized) eigenvector in $\mathcal{P}_1^m$ ($\mathcal{P}_2^m$) and the corresponding eigenvalue is positive and larger than the absolute value of any other eigenvalue. Furthermore, observe that (1.3) is equivalent to the integral equation $\lambda T_{AB}\mathbf{u} = \mathbf{u}$, thus we have the following existence theorem.

THEOREM 3.3. *If $B(x) > 0$ ($\int_x^\beta B(s)\, ds \geqslant 0$ and each component $b_{ij}(x)$ of $B(x)$ is not identically zero) on $[\alpha, \beta]$, then (1.3) has an essentially unique eigenvector whose components are nonnegative (have nonnegative derivatives) on $[\alpha, \beta]$, and the corresponding eigenvalue is positive and smaller than the absolute value of any other eigenvalue.*

4. **Comparison theorems.** For convenience of reference, (1.3) is rewritten here

$$\text{(4.1)} \qquad \begin{cases} \left[A(x)\mathbf{u}^{(n)}\right]^{(n)} + (-1)^{n+1}\lambda B(x)\mathbf{u} = \mathbf{0}, \\ \mathbf{u}^{(i)}(\alpha) = \left[A\mathbf{u}^{(n)}\right]^{(i)}(\beta) = \mathbf{0}, \qquad i = 0, 1, \ldots, n-1, \end{cases}$$

together with two other systems

$$\text{(4.2)} \qquad \begin{cases} \left[C(x)\mathbf{w}^{(n)}\right]^{(n)} + (-1)^{n+1}\mu B(x)\mathbf{w} = \mathbf{0}, \\ \mathbf{w}^{(i)}(\alpha) = \left[C\mathbf{w}^{(n)}\right]^{(i)}(\beta) = \mathbf{0}, \qquad i = 0, 1, \ldots, n-1, \end{cases}$$

$$\text{(4.3)} \qquad \begin{cases} \left[C(x)\mathbf{v}^{(n)}\right]^{(n)} + (-1)^{n+1}\Lambda D(x)\mathbf{v} = \mathbf{0}, \\ \mathbf{v}^{(i)}(\alpha) = \left[C(x)\mathbf{v}^{(n)}\right]^{(i)}(\beta) = \mathbf{0}, \qquad i = 0, 1, \ldots, n-1, \end{cases}$$

where $C(x)$, $D(x)$ satisfy the same assumptions as those for $A(x)$, $B(x)$ respectively.

Let T_{CB}, T_{CD} denote the compact operators on $\mathfrak{B}^m$ defined by

$$T_{CB}\mathbf{u}(x) = \int_\alpha^\beta G_C(x, s)B(s)\mathbf{u}(s)\, ds, \quad T_{CD}\mathbf{u}(x) = \int_\alpha^\beta G_C(x, s)D(s)\mathbf{u}(s)\, ds$$

respectively. Also let λ_0, Λ_0 be, respectively, the smallest positive eigenvalues of (4.1) and (4.3).

THEOREM 4.1. *Suppose $B(x) > 0$, $\int_x^\beta D(s)\, ds \geqslant \int_x^\beta B(s)\, ds$ and $\int_\alpha^x a_i^{-1}(s)\, ds \leqslant \int_\alpha^x c_i^{-1}(s)\, ds$ $(1 \leqslant i \leqslant m)$ on $[\alpha, \beta]$. Then $\Lambda_0 \leqslant \lambda_0$ and equality holds if, and only if, $A(x) \equiv C(x)$ and $D(x) \equiv B(x)$.*

PROOF. By Lemma 2.1, for any $\mathbf{u} \in \mathcal{P}_1^m$,

$$\int_\alpha^\beta G_A(x, s)B(s)\mathbf{u}(s)\, ds \leqslant \int_\alpha^\beta G_C(x, s)\, B(s)\mathbf{u}(s)\, ds.$$

Thus $T_{AB}\mathbf{u} \leqslant T_{CB}\mathbf{u}$ for $\mathbf{u} \in \mathcal{P}_1^m$. Moreover by Theorem 3.1, T_{AB} and T_{CB} are both u_0-positive with respect to $\mathcal{P}_1^m$. According to the abstract comparison theorem for u_0-positive operators [3, 1], we have $\mu_0 \leqslant \lambda_0$ where μ_0 is the smallest positive eigenvalue of (4.2), and $\mu_0 = \lambda_0$ if and only if their corresponding eigenvectors are dependent, which is impossible unless $A(x) \equiv C(x)$. Similarly, $T_{CB}\mathbf{u} \leqslant T_{CD}\mathbf{u}$ for $\mathbf{u} \in \mathcal{P}_2^m$, and T_{CB}, T_{CD} are u_0-positive with respect to $\mathcal{P}_2^m$. The same argument establishes $\Lambda_0 \leqslant \mu_0$ and thus $\Lambda_0 \leqslant \lambda_0$. Q.E.D.

Utilizing Lemma 2.4, Theorem 3.2, and constructing appropriate operator T_{AD}, the following theorem can be similarly established.

THEOREM 4.2. *Suppose* $D(x) > 0$, $\int_x^\beta D(s)\,ds \geqslant \int_x^\beta B(s)\,ds \geqslant 0$ *and* $\int_\alpha^x a_i^{-1}(s)\,ds \leqslant \int_\alpha^x c_i^{-1}(s)\,ds$ $(1 \leqslant i \leqslant m)$ *and each element of* B *is nonzero on* $[\alpha, \beta]$. *Then* $\Lambda_0 \leqslant \lambda_0$ *and equality holds if, and only if,* $A(x) \equiv C(x)$ *and* $B(x) \equiv D(x)$.

THEOREM 4.3. *Suppose* $\int_x^\beta D(s)\,ds \geqslant \int_x^\beta B(s)\,ds \geqslant 0$, $0 < c_i(x) \leqslant a_i(x)$ $(1 \leqslant i \leqslant m)$ *and each element of* B *is nonzero on* $[\alpha, \beta]$. *Then* $\Lambda_0 \leqslant \lambda_0$ *and equality holds if, and only if,* $A(x) \equiv C(x)$ *and* $B(x) \equiv D(x)$.

We have thus generalized the results given in [1]. Such generalizations may seem to be superfluous were it not that comparison of eigenvalues is also possible for nonselfadjoint equations exemplified by the following

or equivalently,

$$(ru_1'')'' + \lambda((ru_1)'' + 2ru_1'' - pu_1) + 2\lambda^2 ru_1 = 0,$$
$$u_1(\alpha) = u_1''(\alpha) = 0 = u_1'(\beta) = (ru_1'')'(\beta).$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'' + \lambda \begin{bmatrix} 1 & 1/r(x) \\ p(x) & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{0} \qquad (r(x) > 0),$$
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(\alpha) = \mathbf{0} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'(\beta)$$

5. **Concluding remarks.** Close examination of the preceding development reveals the fact that differential operators, whose Green's functions satisfy "positivity" properties similar to those in §2, are essential to the comparison of eigenvalues. We mention here one other example without proof:

$$\begin{cases} L_a[u] = (-1)^{n-k}\left[a(x)u^{(k)}\right]^{(n-k)}, \\ u^{(i)}(\alpha) = 0 = \left[au^{(k)}\right]^{(i)}(\beta), \qquad i = 0, 1, \ldots, k-1. \end{cases}$$

Furthermore, systems of the form (3.1) seem to be too restrictive in view of the proof of Theorem 3.1. To illustrate this we will only confine ourselves to citing the following

$$\begin{cases} \left[A(x)\mathbf{u}^{(k)}\right]^{(n-k)} - (-1)^{n-k}\lambda \sum_{i=0}^{k-1} B_i(x)\mathbf{u}^{(i)} = \mathbf{0}, \\ \mathbf{u}^{(i)}(\alpha) = \mathbf{0} = \left[A\mathbf{u}^{(k)}\right]^{(i)}(\beta), \qquad i = 0, 1, \ldots, k-1, \end{cases}$$

where $A(x)$ is an $m \times m$ diagonal matrix whose elements are positive functions of class $C^{(n-k)}[\alpha, \beta]$ and $B_i(x)$ are matrices of continuous functions [4].

References

1. C. C. Travis, *Comparison of eigenvalue for linear differential equations of order* $2n$, Trans. Amer. Math Soc. **177** (1973), 363–374. MR **47** #5356.

2. M. A. Krasnosel'skiĭ, *Positive solutions of operator equations*, Fizmatgiz, Moscow, 1962; English Transl., Noordhoff, Groningen, 1964, MR **26** #2862; **31** #6107.

3. K. Kreith, *Oscillation theory*, Lecture Notes in Math., vol. 324, Springer-Verlag, Berlin and New York, 1973.

4. R. D. Gentry and C. C. Travis, *Comparison of eigenvalues associated with linear differential equations of arbitrary order*, Trans. Amer. Math. Soc. **223** (1976), 167–179.

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