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A CHARACTERIZATION OF CERTAIN COMPACT FLOWS[1]

RONALD A. KNIGHT

ABSTRACT. Compact flows on certain 2-manifolds are characterized in terms of the bilateral stability properties of the orbits.

1. **Introduction.** Beck and Wu have obtained topological characterizations of compact flows on certain surfaces [1], [7]. The major result of this paper is the characterization of such closed flows on certain 2-manifolds in terms of the flow structure itself. In particular, compact flows are characterized by the stability properties of the periodic orbits and critical points.

A pair (X, π) consisting of a topological space X and a continuous mapping $\pi: X \times R \to X$, from the product space $X \times R$ into X, where R denotes the real numbers, is called a *dynamical system* or (continuous) *flow* whenever $\pi(x, 0) = x$ for each x in X, $\pi(\pi(x, t), s) = \pi(x, t + s)$ for each x in X and t and s in R, and π is continuous on $X \times R$. An orientable 2-manifold X is said to be *dichotomic* if every simple closed curve C in X decomposes X into two connected sets with common boundary C. The phase spaces considered in this paper are dichotomic.

We shall denote the trajectory (orbit) through x, orbit closure of x, limit set of x, and prolongation of x by $C(x)$, $K(x)$, $L(x)$, and $D(x)$, respectively. For basic properties of dynamical systems we refer the reader to [2] and [3].

A flow is of *characteristic* 0 if and only if $D(x) = K(x)$ for each x in the phase space. A flow is called *compact* or *periodic* provided each of its orbits is compact. Throughout the paper we shall use P and S to denote the sets of periodic and critical points of a given flow, respectively. Note that an orbit is compact if and only if it is in $P \cup S$.

The extension of a flow (X, π) to the one point compactification X^* of X is denoted by (X^*, π^*). For each function on X we shall denote its extension to X^* with the superscript *. For a given flow we use S^* to denote the critical points of the extended flow.

2. **Preliminary results.**

LEMMA 1. *Let (X, π) be a compact flow with a dichotomic phase space. If S_0 is a compact component of S, then S_0 is bilaterally stable.*

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PROOF. For each point x in $X - S$, $C(x)$ is bilaterally stable according to the Cycle Stability Theorem [3, 3.3, p. 196] yielding $D(x) = D(C(x)) = C(x)$ [2, 7.6, p. 77]. Thus, $D(X - S) = X - S$ and $D(S) = S$. For a point s_0 in S_0 the set $D(s_0) \cap S_0$ is compact and contains a compact component of $D(s_0)$. Thus, $D(s_0) = D(s_0) \cap S_0$ [2, 6.12.2, p. 68], and hence, $D(S_0) = S_0$. The proof is complete.

PROPOSITION 2. *Let (X, π) be a compact flow and X be dichotomic. If S is totally disconnected, then (X, π) is of characteristic* 0.

PROOF. According to the Cycle Stability Theorem and Lemma 1, each trajectory is bilaterally stable. Hence, $D(x) = D(C(x)) = C(x) = K(x)$ for each point x in X.

The only planar compact flows of characteristic 0 are critical flows and flows consisting of a global Poincaré center [5].

PROPOSITION 3. *Let (X, π) be a compact flow with dichotomic phase space and let S be totally disconnected. Then* (i) *if S is empty, X is an open annular region*; (ii) *if S is nonempty, one of the following holds*:

(a) *X is homeomorphic to R^2 and S consists of a global Poincaré center*; *or*

(b) *X is homeomorphic to S^2 and S consists of two Poincaré centers s_1 and s_2 such that s_i is a global Poincaré center relative to $X - \{s_j\}$, $i \neq j$.*

PROOF. The torus, Mobius strip, and Klein bottle are not dichotomic spaces so that Proposition 6.2 of [7] applies, and hence, $X - S$ consists of the disjoint union of open annuli. If $S = \varnothing$, the connectedness of X implies that X consists of exactly one annular region. If $S \neq \varnothing$, S consists of Poincaré centers since (X, π) is of characteristic 0 [4, 10]. Each point of S is in the boundary of exactly one annular region. If S consists of one Poincaré center, then the connectedness of X implies that X contains a single annular region, and hence, is homeomorphic to R^2. Suppose that S consists of at least two points. Let x be an element of S and let H be the annular region whose boundary contains x. If no point of $S - \{x\}$ is in $\overline{H}$, then $\overline{H}$ and $X - \overline{H}$ are disjoint closed sets. This is clearly impossible since X is connected. Hence, there is a point y in $S \cap \overline{H}$ distinct from x. The set $\overline{H}$ must be $H \cup \{x, y\}$ and $\overline{H}$ is homeomorphic to S^2. If $S \neq \{x, y\}$, then $\overline{H}$ and $X - \overline{H}$ are disjoint closed sets which is absurd. Hence, $X = \overline{H}$ and X is homeomorphic to R^2.

3. **Characterization.** Seibert and Tulley showed [6] that a planar flow is compact if and only if it is Poisson stable at each point. Hajek extended the result to dichotomic phase spaces [3, 1.6, p. 183]. According to Beck [1] the set of periodic points of a planar compact flow is composed of countably many open periodic annular regions. He thus characterized such flows in terms of the complement of their critical points. Wu obtained similar results [7] for certain 2-manifolds. In the following theorem we characterize compact flows on a dichotomic phase space in terms of the bilateral stability of S^* and the orbits in P.

THEOREM 4. *A flow on a dichotomic phase space X is compact if and only if each component of the boundary of S^* and each periodic orbit are bilaterally stable in X^*.*

PROOF. Throughout the proof we shall refer to the topology on X^*. Let (X, π) be compact and let K be a component of the boundary of S^*. The component S_0 of S which contains K is bilaterally stable by Lemma 1, and hence, δS_0 and K are bilaterally stable [3, 4.13.4, p. 114]. Furthermore, each periodic orbit is bilaterally stable by the Cycle Stability Theorem.

Conversely, we first show that P is open. Let $C(x)$ be any periodic orbit. Since $C(x)$ is bilaterally stable and S^* is closed there is a compact neighborhood V of $C(x)$ disjoint from S^*. Let y be any point of V. Then either $L^*(y) \subset P$ or, for each $z \in L^*(y)$, we have $L^*(z) \subset S^*$ [3, 1.11, p. 184]. Since $V \cap S^* = \varnothing$, we have $L^*(y) \subset P$. Suppose y is not periodic and let z be any point of $L^*(y)$. Then $y \notin L^*(y)$ and there is a compact invariant neighborhood U of the periodic orbit $C(z)$ excluding y. In fact, there is a compact invariant neighborhood W of $C(z)$ contained in the interior of U. But this means that $L^*(y) \subset X^* - W$, and hence, that $z \notin L^*(y)$, which is absurd. Thus, $y \in L^*(y)$ which means that y is periodic and $C(x) \subset V \subset P$. Hence, P is open.

Next, we let $x \in X^* - P \cup S^*$. Then either $L^*(x) \subset P$ or $L^*(y) \subset S^*$ for each y in $L^*(x)$. Since P is open, $L^*(x) \not\subset P$. The bilateral stability of each component of δS^* implies that δS^*, and hence, S^* are bilaterally stable. There is an open invariant neighborhood V of S^* whose closure excludes x. Thus, $L^*(x) \subset X^* - V$ so that $L^*(y) \subset X^* - V$ for each y in $L^*(x)$ which means, of course, that $L^*(y) \not\subset S^*$ for any y in $L^*(x)$. This contradiction yields $X^* = P \cup S^*$.

COROLLARY 4.1. *A flow on a dichotomic space X is compact if and only if S^* and each periodic orbit are bilaterally stable in X^*.*

COROLLARY 4.2. *The only compact flows on a dichotomic phase space having each orbit bilaterally stable are those where $X = P$, $X = S$, S consists of a global center, or $X = S^2$ and S consists of two centers.*

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MATHEMATICS DIVISION, NORTHEAST MISSOURI STATE UNIVERSITY, KIRKSVILLE, MISSOURI 63501