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# ON A GROUP THAT CANNOT BE THE GROUP OF A 2-KNOT[1]

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ABSTRACT. It is proved that a homomorph of the group of trefoil knot cannot be the group of a 2-knot in 4-sphere.

In this short note we will give a negative answer to a problem proposed by Fox [2, Problem 28]. Fox asked if a finitely presented group that is a homomorph of a knot group and is infinite cyclic over its commutator subgroup is the group of a locally flat 2-sphere in $R^4$. We prove

THEOREM. *$G = (a, b\colon a^2b^{-3}, [a, b]^2)$ cannot be the group of a locally flat n-sphere in $(n + 2)$-sphere $S^{n+2}$ for $n \geqslant 2$, where $[a, b] = aba^{-1}b^{-1}$.*

Since $G/G'$ is infinite cyclic and $G$ is a factor group of $G^* = (a, b\colon a^2b^{-3})$ that is the group of the trefoil knot, our theorem gives a negative answer to the problem, where $G'$ denotes the commutator subgroup of $G$.

Our proof is based on the following

PROPOSITION 1 [4, p. 106]. *If a finitely presented group $\pi$ is the group of a locally flat n-sphere $K^n$ in $S^{n+2}$, $n \geqslant 1$, i.e. $\pi \cong \pi_1(S^{n+2} - K^n)$, then*

(1) *$\pi/\pi'$ is infinite cyclic,*

(2) *the weight of $\pi$ is one, and*

(3) *$H_2(\pi) = 0$, where $H_2(\pi)$ denotes the second homology group of $\pi$ with integral coefficient and trivial action of $\pi$ on the coefficient group.*

*Furthermore, if $n \geqslant 3$, then* (1), (2) *and* (3) *are sufficient conditions for $\pi$ to be the group of some n-knot $K^n$ in $S^{n+2}$.*

Our group $G$ satisfies (1) and (2), since $G^*$ does as the knot group. Therefore, to prove the theorem we must show

PROPOSITION 2. $H_2(G) \neq 0$.

PROOF. Let $F$ be the free group with two free generators $a$, $b$. For a set, $X$, of elements of $F$, we denote by $X^F$ the normal closure of $X$ in $F$.

Now, let $R = \{a^2b^{-3}, [a, b]^2\}^F$. Then $G \cong F/R$ and, by Hopf [3], $H_2(G) = R \cap F'/[R, F]$. Since $[a, b]^2 \in F'$, it suffices to show that $[a, b]^2 \neq 1$ in the group

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$$H = (a, b\colon [a^2b^{-3}, a], [a^2b^{-3}, b], [[a,b]^2, a], [[a,b]^2, b]).$$

Let $S = \{a^2ba^{-2}b^{-1}, ab^3a^{-1}b^{-3}, [[a,b]^2, a], [[a,b]^2, b]\}$. Obviously, $H \cong F/S^F$.

Consider the following sets.

$$T = \{a^2, b^3, [a,b]^2[b,a^{-1}]^{-2}, [a,b]^2[b^{-1},a]^{-2}\},$$

$$U = \{a^2, b^3, (ab)^3(ba)^{-3}\},$$

$$V = \{a^2, b^3, [a,b]^2\}.$$

We claim now

LEMMA 1. $T^F = U^F$.

PROOF. Probably the simplest way to prove this is to show that
(1) $(ab)^3(ba)^{-3} = 1$ in $F/T^F$ and
(2) $[a,b]^2[b,a^{-1}]^{-2} = 1$, $[a,b]^2[b^{-1},a]^{-2} = 1$ in $F/U^F$, and direct computation works as follows:

$$\begin{aligned} b^{-2}ab^{-1}\{(ab)^3(ba)^{-3}\}ba^{-1}b^2 &= [b,a^{-1}]^2[a,b^{-1}]^2 \\ &= [a,b]^2[a,b]^{-2} = 1 \quad \text{in } F/T^F, \end{aligned}$$

and

$$\begin{aligned} [a,b]^2[b,a^{-1}]^{-2} &= ab(ab^{-1}aba)b^{-1}(abab^{-1}a)bab^{-1} \\ &= ab(babab^{-1}ab^{-1})b^{-1}(b^{-1}ab^{-1}abab)bab^{-1} \\ &= ab^{-1}abababab^{-1}ab^{-1} = 1 \quad \text{in } F/U^F, \end{aligned}$$

and similarly,

$$[a,b]^2[b^{-1},a]^{-2} = 1 \quad \text{in } F/U^F.$$

LEMMA 2. $(ab)^3(ba)^{-3} \in V^F$.

PROOF. Again it is enough to show that $(ab)^3(ba)^{-3} = 1$ in $F/V^F$. In fact,

$$\begin{aligned} (ab)^3(ba)^{-3} &= abab(abab^{-1})ab^{-1}ab^{-1} \\ &= abab(bab^{-1}a)ab^{-1}ab^{-1} \\ &= abab^{-1}abab^{-1} = 1 \quad \text{in } F/V^F. \end{aligned}$$

Now we return to the proof of Proposition 2.

To lead a contradiction, we suppose $[a,b]^2 = 1$ in $H$. Then $[a,b]^2$ is contained in $S^F$ and hence $[a,b]^2$ is in $T^F$. By Lemma 1, then, $[a,b]^2$ is in $U^F$. Lemma 2 now yields $U^F = V^F$. This implies that two groups $G_1 = F/U^F$ and $G_2 = F/V^F$ are isomorphic. It is impossible, however, because $G_1 =$

$(a, b: a^2, b^3, (ab)^3(ba)^{-3})$ is denoted by $\langle 2, 3|3\rangle$ in [1, p. 76] and its order is 48 [1, p. 77]. On the other hand, $G_2 = (a, b: a^2, b^3, [a, b]^2)$ has order 24 [1, p. 134].

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