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MACKEY'S THEOREM FOR NONUNITARY REPRESENTATIONS

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ABSTRACT. We present an analog to the Intertwining Number Theorem of Mackey in a setting which arises naturally in the study of the representation theory of p-adic linear groups.

Introduction. It is a well-known result of Mackey [2] that the space of operators intertwining two unitarily induced representations of a locally compact group may, under certain circumstances, be decomposed into a sum of spaces of operators intertwining certain conjugate representations of the inducing representations. In our work on admissible representations of p-adic linear groups [1] we have found it necessary to define a kind of nonunitary induction and to make use of an analogue of the theorem of Mackey cited above.

Since both the type of induction we employ and the conditions we impose occur naturally in a wide variety of circumstances, it is to be hoped that the above-mentioned analogue, for which we here furnish a proof, may result in the application of "Mackey Theory" to groups to which it has so far not been applied.

1. Let G be a topological group, let H, H_1, H_2 be closed subgroups of G and let τ, τ_1, τ_2 be, respectively, (algebraic) representations of H, H_1, H_2 on complex vector spaces W, W_1, W_2. If x is any element of G, we define the *conjugate representation* τ^x of τ to be the representation of $x^{-1}Hx$ on V given by $\tau^x(x^{-1}hx) = \tau(h)$ for h in H. If A is a linear transformation from W_1 to W_2 such that $A\tau_1(g) = \tau_2(g)A$ for all g in $H_1 \cap H_2$ then we say that A *intertwines* τ_1 with τ_2. We denote the space of all such intertwining operators by $I(\tau_1, \tau_2)$.

The notion of induction mentioned in the introduction may be described as follows. Let W^G be the space of all W-valued functions f on G which satisfy the condition that $f(hx) = \tau(h)f(x)$ for h in H, x in G. Let W_c^G be the subspace of functions in W^G which are supported on sets whose images in the right coset space $H \setminus G$ are compact. Then we define the representation *induced by* τ *on* G to be the representation τ^G of G on W^G given by $(\tau^G(g)f)(x) = f(xg)$ for x, g in G and the representation *induced compactly*

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(c-induced) by τ on G to be the restriction T_c^G of T^G to the G-subspace W_c^G of W^G.

We may now state our version of Mackey's Theorem.

THEOREM. *Let G be a topological group; let H be a closed subgroup of G and let K be a subgroup of G which contains an open compact subgroup. For some indexing set Λ, let x_l, l in Λ be a complete set of (H, K)-double cosets in G. Let σ be a representation of K on a finite dimensional vector space V and let τ be a representation of H on a vector space W of arbitrary dimension. Then there is a vector space isomorphism of $I(\sigma_c^G, \tau^G)$ onto the direct product of the spaces $I(\sigma, \tau^{x_l})$, l in Λ. If, in addition, the image of K in $H \setminus G$ is compact, then the above-mentioned isomorphism maps $I(\sigma_c^G, \tau^G)$ onto the direct sum of the spaces $I(\sigma, \tau^{x_l})$, l in Λ.*

PROOF. Following Mackey [2], we define the space of (σ, τ)-*spherical functions*, $S(\sigma, \tau)$, to be the set of all functions s from G to $\mathrm{End}_{\mathbf{C}}(V, W)$ for which $s(hxk) = \tau(h)s(x)\sigma(k)$ for h in H, k in K, x in G. As above, we define the set of (σ, τ)-c-spherical functions to be the subspace $S_c(\sigma, \tau)$ consisting of functions whose supports have compact image in $H \setminus G$. We first show that there is an isomorphism of $I(\sigma_c^G, \tau^G)$ onto $S(\sigma, \tau)$ which maps $I(\sigma_c^G, \tau_c^G)$ onto $S_c(\sigma, \tau)$.

To this end, we define for each v in V the function f_v from G to V by setting $f_v(x) = \sigma(x)v$ for x in K, $f_v(x) = 0$ otherwise. We note that f_v is in V_c^G for all v in V. If A is an element of $I(\sigma_c^G, \tau^G)$ we define the function s_A from G to $\mathrm{End}_{\mathbf{C}}(V, W)$ by $s_A(x)(v) = A(f_v)(x)$ for x in G, v in V. Then one checks that s_A is an element of $S(\sigma, \tau)$.

Now let s be an element of $S(\sigma, \tau)$ and let f be in V_c^G. Then for fixed x in G the function $s(xz^{-1})f(z)$ is constant on right cosets of K and is supported on some finite union of these cosets. Thus, if Z is a set of right coset representatives of K in G, we may define the map A_s from V_c^G to W^G by setting $(A_sf)(x) = \sum_{z \text{ in } Z} s(xz^{-1})f(z)$. As above, one checks that A_s is a well-defined element of $I(\sigma_c^G, \tau^G)$ and one may verify that the maps $A \to s_A$ and $s \to A_s$ are inverse to each other.

Finally, if A is in $I(\sigma_c^G, \tau_c^G)$ then since V is finite dimensional there exists a subset F of G whose image in $H \setminus G$ is compact such that for all v in V, the support of $A(f_v)$ is contained in F. Thus s_A is supported on F so that s_A is in $S_c(\sigma, \tau)$. Conversely, if s is in $S_c(\sigma, \tau)$ and f is in V_c^G then s is supported on some subset E of G whose image in $H \setminus G$ is compact and there exists a finite subset Z_0 of Z so that $f(z) = 0$ for z in $Z - Z_0$. Hence A_sf is supported on the set $\bigcup_{z \text{ in } Z_0} Ez$ whose image in $H \setminus G$ is compact and we see that A_s is in $I(\sigma_c^G, \tau_c^G)$. The map $A \to s_A$ has now been shown to be an isomorphism with the desired properties.

Now for l in Λ let S_l be the subspace of S consisting of functions whose supports lie in Hx_lK. We note that $S(\sigma, \tau)$ is the direct product of the spaces S_l; that if K has compact image in $H \setminus G$ then S_l lies in $S_c(\sigma, \tau)$ for l in Λ and that, in fact, $S_c(\sigma, \tau)$ is the direct sum of the spaces S_l. All that remains to be shown, then, is that, for l in Λ, S_l is isomorphic to $I(\sigma, \tau^{x_l})$.

However, if s is in S_l and k is in $K \cap x_l^{-1}Hx_l$, then $s(x_l)\sigma(k) = s(x_lk) = \tau(x_lkx_l^{-1})s(x_l)$ so that $s(x_l)$ is in $I(\sigma, \tau^{x_l})$. Conversely, let α be in $I(\sigma, \tau^{x_l})$ and let x be in Hx_lK. Suppose $x = h_1x_lk_1 = h_2x_lk_2$, h_i in H, k_i in K, $i = 1, 2$. Then $k_1k_2^{-1} = x_l^{-1}h_1^{-1}h_2x_l$ lies in $K \cap x_l^{-1}Hx_l$ so that $\alpha\sigma(k_1k_2^{-1}) = \tau(h_1^{-1}h_2)\alpha$ and hence $x \mapsto \tau(h_1)\alpha\sigma(k_1)$ is a well-defined function of x. If we now define s_α on G by $s_\alpha(x) = \tau(h)\alpha\sigma(k)$ for $x = hx_lk$ in Hx_lK, $s_\alpha(x) = 0$ otherwise, then it may be checked that s_α is in S_l; that $s_\alpha(x_l) = \alpha$ and that $s_{s(x_l)} = s$. Thus the map $s \mapsto s(x_l)$ is an isomorphism of S_l onto $I(\sigma, \tau^{x_l})$, which was to be shown.

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