LOWER BOUNDS FOR THE ZEROS OF BESSEL FUNCTIONS

ROGER C. MCCANN

Abstract. Let $j_{p,n}$ denote the $n$th positive zero of $J_p$, $p > 0$. Then

$$j_{p,n} > (j_{0,n}^2 + p^2)^{1/2}.$$  

We begin by considering the eigenvalue problem

(1) $-(xy')' + x^{-1}y = \lambda^2 x^{2p-1}y$, \quad $\lambda, p > 0$,

(2) $y(a) = y(1) = 0$, \quad $0 < a < 1$.

For simplicity of notation we will set $q = p^{-1}$. It is easily verified that the general solution of (1) is

$$y(x) = C_1 J_q(\lambda q x^{1/q}) + C_2 Y_q(\lambda q x^{1/q})$$

and that the eigenvalues are given by

$$J_q(\lambda q) Y_q(\lambda q a^{1/q}) - J_q(\lambda q a^{1/q}) Y_q(\lambda q) = 0.$$  

If $z_n(a, r)$ denotes the $n$th positive zero of $J_r(z) Y_r(z a^{1/q}) - J_r(z a^{1/q}) Y_r(z) = 0$, then the $n$th eigenvalue, $\lambda^2_n(a)$, of (1), (2) is given by

(3) $\lambda^2_n(a) = \left( z_n(a, q)/q \right)^2$.

Let $j_{r,n}$ denote the $n$th positive zero of $J_r$. On p. 38 of [4] it is shown that $z_n(a, r) \to j_{r,n}$ as $a \to 0^+$ whenever $r$ is a positive integer. The restriction on $r$ is extrinsic so that

(4) $\lim_{a \to 0^+} z_n(a, r) = j_{r,n}$, \quad $r > 0$.

Let $R[p, y]$ denote the Rayleigh quotient

$$R[p, y] = \int_a^1 \left( -(xy')' + x^{-1}y \right) y \, dx / \int_a^1 x^{2p-1}y^2 \, dx.$$  

It is well known that the eigenvalues \{$\lambda^2_n(p)$\} of (1), (2) can be obtained from the Rayleigh quotient [5]. Let $V$ denote the linear space of all functions in $C^2((a, 1))$ which satisfy the boundary conditions (2). Then

$$\lambda^2_n(p) = \min_{y \in V, y \neq 0} R[p, y].$$

Let $y_1, y_2, \ldots, y_n$ be $n$ functions in $V$, $A$ denote the subspace of $V$ spanned by $y_1, y_2, \ldots, y_n$ and $A^\perp$ denote the orthogonal complement of $A$ relative to $V$. Then

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where the maximum is taken over all sets of \( n \) functions in \( V \).

Whenever \( p > 0 \) we have that \( x^{2p-1} < x^{-1} \) for all \( x \in (0, 1) \). Then

\[
R[p, y] = \frac{\int_a^1 (xy')' \,dx}{\int_a^1 x^{2p-1} y^2 \,dx} = \frac{\int_a^1 x^{2p-1} y^2 \,dx}{\int_a^1 x^{2p-1} y^2 \,dx} \geq Q[p, y] + 1,
\]

where

\[
Q[p, y] = \int_a^1 - (xy')' \,dx / \int_a^1 x^{2p-1} y^2 \,dx
\]

is the Rayleigh quotient for the eigenvalue problem

\[
-(xy')' = \mu^2 x^{2p-1} y,
\]

\[
y(a) = y(1) = 0,
\]

Equation (6) is equivalent to

\[
x^2 y'' + y' + \mu^2 x^{2p-1} y = 0.
\]

It is easily checked that the general solution of (8) and, hence, of (6) is (recall that \( q = p^{-1} \))

\[
y(x) = C_1 J_0(\mu q x^{1/q}) + C_2 Y_0(\mu q x^{1/q})
\]

and that the eigenvalues are given by

\[
J_0(\mu q) Y_0(\mu q a^{1/q}) - J_0(\mu q a^{1/q}) Y_0(\mu q) = 0.
\]

In particular the \( n \)th eigenvalue, \( \mu_n^2(a) \), of (6), (7) is given by

\[
\mu_n^2(a) = (z_n(a, 0)/q)^2.
\]

From (3), (5), and (9) we obtain

\[
(z_n(a, q)/q)^2 > (z_n(a, 0)/q)^2 + 1.
\]

If we now replace \( q \) by \( p \), let \( a \to 0^+ \) in (10), and using (4) we find that

\[
(j_{p,n}/p)^2 > (j_{0,n}/q)^2 + 1.
\]

**Theorem.** \( j_{p,n} > ((j_{0,n})^2 + p^2)^{1/2} \) whenever \( p > 0 \).

**Corollary.** \( j_{p,n} > ((n - \frac{1}{4})^2 + p^2)^{1/2} \) whenever \( p > 0 \).

**Proof.** It is known (see [9, p. 489]) that the positive zeros of \( J_0 \) lie in the intervals \( (m\pi + \frac{3}{4} \pi, m\pi + \frac{5}{4} \pi) \) for \( m = 0, 1, 2, \ldots \). Hence, \( j_{0,n} > (n - 1)^2 + \frac{3}{4} \pi = (n - \frac{1}{4})^2 \). The desired result follows.

In [8] it is shown that

\[
j_{p,n} = p + a_n p^{1/3} + b_n p^{-1/3} + O(p^{-1}) \quad (n = 1, 2, \ldots),
\]

where \( a_n \) and \( b_n \) are independent of \( p \). Hence,

\[
j_{p,n} = p^2 + c_n p^{4/3} + O(p^{2/3}) \quad (n = 1, 2, \ldots),
\]

where \( c_n \) is independent of \( p \). This shows that the second term of the lower
bound for $j_{p,n}$ given in the Theorem is of the wrong order. Other asymptotic expansions for $j_{p,n}$ may be found in [1], [2], and [6].

In [3] it is shown that for $0 \leq p \leq \frac{1}{2}$

$$p\pi/2 + \left(n - \frac{1}{2}\right)\pi \leq j_{p,n}. \tag{11}$$

For $p = 0$ the result of the Theorem is exact, while the expression in (11) has a strict inequality. Hence, our result is stronger than (11) whenever $p$ is sufficiently small. However, when $p = \frac{1}{2}$, the result in (11) is exact. Hence, for $0 \leq p \leq \frac{1}{2}$ neither result implies the other. It should be emphasized that the Theorem is valid for all $p > 0$, while (11) is valid only for $0 \leq p \leq \frac{1}{2}$.

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REFERENCES


DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106

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