EXTENSIONS OF TOTALLY PROJECTIVE GROUPS
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Abstract. It is an unpublished observation of L. Fuchs and E. A. Walker that if \( L \) is a fully invariant subgroup of the totally projective \( p \)-group \( G \), then both \( L \) and \( G/L \) are totally projective. In this note we treat the more difficult converse question.

It is an unpublished observation of L. Fuchs and E. A. Walker that if \( L \) is a fully invariant subgroup of the totally projective \( p \)-group \( G \), then both \( L \) and \( G/L \) are totally projective. In this note we treat the more difficult converse question.

Recall that a reduced abelian \( p \)-group \( G \) is said to be totally projective if \( p^\alpha \text{Ext}(G/p^\alpha G, K) = 0 \) for all abelian groups \( K \) and all ordinals \( \alpha \). There are numerous characterizations of totally projective groups more enlightening than the definition [1, p. 90], but the main fact that we need is the following observation: If \( G \) is totally projective, then so are \( p^\alpha G \) and \( G/p^\alpha G \) for all ordinals \( \alpha \); and, conversely, if there is an ordinal \( \alpha \) such that both \( p^\alpha G \) and \( G/p^\alpha G \) are totally projective, then \( G \) is totally projective. It is, of course, this latter statement which we shall generalize.

We shall also require at a crucial juncture another characterization of totally projectives which may be viewed as a generalization of the classical Kulikov criterion. For this we need certain definitions. If \( \lambda \) is a limit ordinal, we call a \( p \)-group \( G \) a \( C_\lambda \)-group provided \( G/p^\alpha G \) is totally projective for all \( \alpha < \lambda \). Thus every abelian \( p \)-group is a \( C_\omega \)-group. Recall that the length of a reduced \( p \)-group \( G \) is just the smallest ordinal \( \lambda \) such that \( p^\lambda G = 0 \). We shall say that a reduced abelian \( p \)-group \( G \) is \( \sigma \)-summable if its socle \( G[p] = \{ x \in G : px = 0 \} \) is the ascending union of a sequence of subgroups \( \{ S_n \}_{n<\omega} \) where for each \( n \) there is an ordinal \( \alpha_n \) less than the length of \( G \) such that \( S_n \cap p^\alpha G = 0 \). We can now state

**Theorem 1.** Let \( \lambda \) be a limit ordinal cofinal with \( \omega \). Then a \( p \)-group \( G \) of length \( \lambda \) is totally projective if and only if \( G \) is a \( \sigma \)-summable \( C_\lambda \)-group.

**Proof.** Let \( G \) be a totally projective group having limit length \( \lambda \) cofinal with \( \omega \). Then \( G/p^\alpha G \) is totally projective for all \( \alpha \) and thus \( G \) is a \( C_\lambda \)-group. Since \( G \) has limit length, it is a direct sum of groups of length strictly less than \( \lambda \) [1, p. 97]. As \( \lambda \) is cofinal with \( \omega \), we actually have such a direct decomposition of \( G \) into countably many summands \( G = \bigoplus_{n<\omega} H_n \) where we may take

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\( \alpha_n = \text{length of } H_n \text{ strictly less than } \lambda \) and \( \alpha_n < \alpha_{n+1} \). Taking \( S_n \) to be the socle of \( H_0 + \cdots + H_n \), we see that \( G \) is \( \sigma \)-summable.

Conversely, suppose \( G \) is a \( \sigma \)-summable \( C_\lambda \)-group of length \( \lambda \). There are two distinct cases. First if \( \lambda = \beta + \omega \) for some ordinal \( \beta \), then the \( \sigma \)-summability of \( G \) implies that \( (p^\beta G)[p] \) is the ascending union of subgroups of bounded height (as computed in \( p^\beta G \)) and, hence, \( p^\beta G \) is a direct sum of cyclic groups by the classical Kulikov criterion. But then we have both \( p^\beta G \) and \( G/p^\beta G \) totally projective and, therefore, \( G \) is totally projective.

Finally, assume that \( \lambda \neq \beta + \omega \) for all \( \beta \) and let the \( S_n \)'s and \( \alpha_n \)'s be as in the definition of \( \sigma \)-summability. Since \( \alpha_n + \omega < \lambda \) for all \( n \), we may assume that each \( \alpha_n \) is of the form \( \beta_n + \omega \). Then by a routine argument we can enlarge each \( S_n \) so that it is maximal disjoint in \( G[p] \) from \( p^{\alpha_n} G \) and still maintain the monotonicity of the sequence \( \{ S_n \}_{n < \omega} \). Indeed such an argument occurs in the standard proofs of the Kulikov criterion. Next we form an ascending sequence of subgroups \( \{ G_n \}_{n < \omega} \) where \( G_n \) is maximal in \( G \) with respect to having \( S_n \) as its socle. Then \( G = \bigcup_{n < \omega} G_n \) is totally projective by [3, Proposition 2.5].

Recall that the height \( h_G(x) \) of a nonzero element \( x \) in the reduced \( p \)-group \( G \) is defined to be the least ordinal \( \alpha \) such that \( x \notin p^{\alpha+1} G \). We also set \( h_G(0) = \infty \). Let \( \alpha = (\alpha_i)_{i < \omega} \) be an increasing sequence of ordinals and symbols \( \infty \); that is, for each \( i \), either \( \alpha_i \) is an ordinal or \( \alpha_i = \infty \) and \( \alpha_i < \alpha_{i+1} \) provided \( \alpha_i \neq \infty \). With each such sequence \( \alpha \) we associate the fully invariant subgroup \( G(\alpha) \) of the \( p \)-group \( G \) defined by

\[
G(\alpha) = \{ x \in G : h_G(p^i x) > \alpha_i \text{ for all } i < \omega \}.
\]

If \( G \) is totally projective, then all of its fully invariant subgroups are of this form (see [1, Theorem 67.1 and Exercise 6, p. 101]). Thus we shall satisfy ourselves with showing that if for some sequence \( \alpha \) both \( G(\alpha) \) and \( G/G(\alpha) \) are totally projective, then \( G \) itself is necessarily totally projective. We suspect that there does not exist a \( p \)-primary group \( G \), necessarily not totally projective, that contains a fully invariant subgroup \( L \) not of the form \( G(\alpha) \) such that both \( L \) and \( G/L \) are totally projective.

Our first result will turn out to be a major step towards proving the desired theorem.

**Theorem 2.** Let \( \lambda \) be a limit ordinal cofinal with \( \omega \) and let \( \alpha = (\alpha_i)_{i < \omega} \) be an increasing sequence of ordinals bounded above by \( \lambda \). If \( G \) is a \( C_\lambda \)-group and if \( G(\alpha) \) is totally projective, then \( G \) is totally projective.

**Proof.** Let \( L = G(\alpha) \). First we wish to show that it is enough to prove the theorem in the special case where (1) \( \lambda = \sup \alpha_i \), (2) \( \lambda \) is the length of \( G \), and (3) \( L \) is a direct sum of cyclic groups. Let \( \delta = \sup \alpha_i < \lambda \). It is not difficult to verify that \( p^\omega L = p^\delta G \) and, hence, \( p^\delta G \) is totally projective. Therefore we need only establish that \( G/p^\delta G \) is totally projective. But if \( \delta < \lambda \), this follows from the assumption that \( G \) is a \( C_\lambda \)-group. Hence we may assume that \( \lambda = \delta = \sup \alpha_i \). Then it is easily seen that \( L/p^\omega L = L/p^\lambda G = (G/p^\delta G)(\alpha) \) and, of
course, $L/p^aL$ is a direct sum of cyclic groups since $L$ is totally projective. Hence the desired reduction follows since $G/p^aG$ is a $C_\alpha$-group if $G$ is.

Since now we are assuming $L$ to be a direct sum of cyclic groups, we have $L[p] = \bigcup_{n=1}^{\infty} T_n$ where $T_n \cap p^nL = 0$ and $T_n \subseteq T_{n+1}$ for each $n$. Now clearly $L[p] = (p^aG)^G[p]$ and, hence, we have a direct decomposition $G[p] = S_0 + L[p]$ where $S_0 \cap p^aG = 0$. It follows that $G[p]$ is the monotone union of the sequence of subgroups $(S_n)_{n<\omega}$ where $T_n = S_0 + T_n$ for $n > 1$. Moreover, $S_n \cap p^aG = 0$ for each $n$ since

$$S_n \cap p^aG = S_n \cap (p^aG)[p] = S_n \cap (p^aL)[p] = S_n \cap p^aL = 0.$$  

We conclude that $G$ is $\sigma$-summable and, therefore, Theorem 1 is applicable.

Next we turn to the very special case of our problem where $a_0 = \alpha$ and $a_i = \infty$ for $i > 1$.

**Lemma.** If $L = (p^aG)[p]$ and $G/L$ is totally projective, then $G$ is totally projective.

**Proof.** First note that

$$p^{a+1}G = p(p^aG)/(p^aG)[p] = p^a(G/L)$$

and, hence, $p^{a+1}G$ is totally projective. But this clearly implies that $p^aG$ is totally projective, and it remains only to show that $G/p^aG$ is also totally projective. Since $L \subseteq p^aG$, we have $G/p^aG \cong (G/L)/p^a(G/L)$, and the latter group is totally projective since $G/L$ is.

**Corollary.** If $L = G(\alpha)$ and $G/L$ is totally projective, then $G/pL$ is totally projective.

**Proof.** Observe that $L/pL = p^a(G/pL)[p]$ where $\alpha = a_0$. Then $(G/pL)/(L/pL) \cong G/L$ is totally projective and we need only apply the lemma to $G/pL$.

We are now in position to prove

**Theorem 3.** If $\alpha$ is an increasing sequence of ordinals and symbols $\infty$ such that both $G(\alpha)$ and $G/G(\alpha)$ are totally projective $p$-groups, then $G$ itself is totally projective.

**Proof.** Let $L = G(\alpha)$. If the sequence $\alpha$ contains any symbols $\infty$, then there is a positive integer $n$ such that $p^nL = 0$. But then repeated applications of the above corollary yield the desired conclusion that $G$ is totally projective. Thus we may assume that $\alpha = (\alpha_i)_{i<\omega}$ is an increasing sequence of ordinals and take $\lambda = \sup \alpha_i$. By Theorem 2, it suffices to show that $G$ is a $C_\lambda$-group. Hence we need only verify that $G/p^aG$ is totally projective for all $n$. But since $p^aL \subseteq p^aG$, we have

$$G/p^aG \cong (G/p^aL)/(p^a(G/p^aL))$$

totally projective since each $G/p^nL$ is totally projective by our Corollary above.
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