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ON THE ACCRETIVITY OF THE INVERSE OF AN ACCRETIVE RELATION

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ABSTRACT. If X is a smooth, reflexive, real Banach space such that a relation A in $X \times X$ is accretive iff A^{-1} is accretive, then X is isomorphic to a Hilbert space.

Let X be a real Banach space with dual X^*. A subset $A \subset X \times X$ is said to be accretive if for all $[x_i, y_i] \in A$, $i = 1, 2$, $(y_1 - y_2, f) \geqslant 0$ for some $f \in F(x_1 - x_2)$, where F is the duality map: $F(x) = \{x^* \in X^* | (x, x^*) = \|x\|^2 = \|x^*\|^2\}$. By definition $A^{-1} = \{[x, y] | [y, x] \in A\}$. It is trivial that if X is a real Hilbert space then A is accretive (or monotone) iff A^{-1} is accretive.

THEOREM. *Suppose X is a smooth, reflexive, real Banach space such that A is accretive in $X \times X$ iff A^{-1} is accretive in $X \times X$. Then X is isomorphic to a Hilbert space.*

PROOF. As X is smooth and reflexive, the duality map F is single-valued. Define $\langle x, y \rangle = (x, F(y))$, $x, y \in X$. It follows from the assumption that $\langle x, y \rangle \geqslant 0$ iff $\langle y, x \rangle \geqslant 0$, and so also that

$$\langle x, y \rangle = 0 \Leftrightarrow \langle y, x \rangle = 0 \quad \forall x, y \in X. \tag{1}$$

For any closed subspace M of X define

$$M^{\perp} = \{x \in X | \langle y, x \rangle = 0 \, \forall y \in M\}. \tag{2}$$

By (1) it is seen that M is a closed subspace of X. If $x \in M \cap M^{\perp}$ then $\|x\|^2 = \langle x, x \rangle = 0$, i.e., $x = 0$; and so

$$M \cap M^{\perp} = \{0\}. \tag{3}$$

Let $z \in X$ be arbitrary. Let x_1 be an element in M that minimizes the norm $\|z - x\|$, $x \in M$ (x_1 exists because M is closed and convex and X is reflexive). This implies that $\|z - x_1\| \leqslant \|z - x_1 + y\| \; \forall y \in M$. Lemma 1.1 in [1] now yields $\langle y, z - x_1 \rangle = 0 \; \forall y \in M$, and so $z - x_1 \in M^{\perp}$. It follows that

$$X = M + M^{\perp}, \tag{4}$$

which together with (3) and the fact that $M^{\perp}$ is a closed subspace shows that M has a topological complement. As M was an arbitrary closed subspace of X, Theorem 1 in [2] implies that X is isomorphic to a Hilbert space.

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