SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A CHARACTERIZATION OF METRIC COMPLETENESS

J. D. WESTON

Abstract. A proof is given of a theorem, relevant to fixed-point theory, which implies that a metric space $(X, d)$ is complete if and only if, for each continuous function $h: X \to \mathbb{R}$ bounded below on $X$, there is a point $x_0$ such that $h(x_0) - h(x) < d(x_0, x)$ for every other point $x$.

If $(X, d)$ is a metric space and $h$ is a function $X \to \mathbb{R}$, by a d-point for $h$ we mean a point $x_0$ of $X$ such that, for every other point $x$,

$$h(x_0) - h(x) < d(x_0, x).$$

In terms of this notion, the following theorem gives a necessary and sufficient condition for $(X, d)$ to be complete. The necessity of the condition generalizes the proposition mentioned by Chi Song Wong in his recent note [1] concerning sufficient conditions for the existence of a fixed point for a function $X \to X$. (For some comparable results, and other methods of proof, see the papers listed in [1], especially those of Brønsted and Ekeland.)

Theorem. If the metric space $(X, d)$ is complete then any lower semicontinuous function $X \to \mathbb{R}$ which is bounded below has a d-point. If $(X, d)$ is not complete there is a uniformly continuous function $X \to \mathbb{R}$ which is bounded below but has no d-point.

Proof. Suppose first that $(X, d)$ is complete, and let $h$ be a function $X \to \mathbb{R}$ which is lower semicontinuous (with respect to $d$) and is bounded below. Taking $x_1$ to be any point of $X$, we choose a sequence $\{x_n\}$ in the following way. For each $n$, let

$$c_n = \inf \{ h(x): h(x_n) - h(x) > d(x_n, x) > 0 \}.$$
and let $x_{n+1}$ be a point such that

(1)  
$$h(x_n) - h(x_{n+1}) \geq d(x_n, x_{n+1})$$

and

(2)  
$$h(x_{n+1}) < c_n + n^{-1}.$$  

(If $x_n$ is a $d$-point for $h$ then $x_{n+1}$ must be $x_n$ and $c_n = \infty$.) From (1) it follows that the sequence $\{h(x_n)\}$ is nonincreasing, and that if $m > n$ then

(3)  
$$h(x_n) - h(x_m) \geq d(x_n, x_m).$$

Since the sequence $\{h(x_n)\}$ is bounded below, it is convergent. Hence, by (3) and the assumed completeness, the sequence $\{x_n\}$ is convergent: let $x_0$ be its limit. Now

(4)  
$$h(x_n) - h(x_0) \geq d(x_n, x_0)$$

for every $n$, because if, for some $n$,

$$h(x_n) - h(x_0) < d(x_n, x_0) - \epsilon,$$

where $\epsilon > 0$, then by the lower semicontinuity of $h$ there would be a neighbourhood $U$ of $x_0$ such that

$$h(x_n) - h(x) < d(x_n, x_0) - \epsilon$$

for every $x$ in $U$, and then $m$ could be such that $x_m \in U$ and $d(x_m, x_0) < \epsilon$, so that, contrary to (3),

$$h(x_n) - h(x_m) < d(x_n, x_0) - \epsilon < d(x_n, x_m).$$

If $x_0$ is not a $d$-point for $h$ then, for some $x$,

(5)  
$$h(x_0) - h(x) \geq d(x_0, x) > 0.$$

From (4) (with $n + 1$ in place of $n$) and (2),

$$h(x) \leq h(x_{n+1}) + h(x) - h(x_0) < c_n + n^{-1} + h(x) - h(x_0).$$

Hence, by (5), we can choose $n$ so that $h(x) < c_n$. From (4) and (5), $h(x_n) > h(x)$, so that $x_n \neq x$ and therefore $d(x_n, x) > 0$, and, moreover,

$$h(x_n) - h(x) \geq d(x_n, x).$$

It now follows from the definition of $c_n$ that $h(x) \geq c_n$, and we have a contradiction. Thus $x_0$ is a $d$-point for $h$.

Now suppose that $(X, d)$ is not complete, and let $\{x_n\}$ be a Cauchy sequence (with respect to $d$) which is not convergent. For any point $x$ of $X$, $\{2d(x, x_n)\}$
is a Cauchy sequence in $\mathbb{R}$: let $h(x)$ be its limit. Then $h(x) > 0$, so the function $h$ is bounded below. Also, if $x_0 \in X$,

$$|h(x_0) - h(x)| \leq 2d(x_0, x),$$

so $h$ is uniformly continuous; and

$$\frac{1}{2}(h(x_0) + h(x)) \geq d(x_0, x),$$

so

$$h(x_0) - h(x) \geq d(x_0, x) + \frac{1}{2}(h(x_0) - 3h(x)).$$

Now, by the definition of $h$, $h(x_m) \to 0$ as $m \to \infty$; therefore $3h(x) < h(x_0)$ if $x = x_m$ and $m$ is large. Thus $x_0$ is not a $d$-point for $h$.

**Remarks.**

(i) When $X = \mathbb{R}$, and $d$ is the usual metric for $\mathbb{R}$, a function $A : X \to \mathbb{R}$ which is uniformly continuous but not bounded below may have, but need not have, a $d$-point.

(ii) When $d$ and $h$ are given, a relation $\ll$ can be defined on $X$ by the stipulation that $x \ll y$ if and only if $h(y) - h(x) \geq d(x, y) > 0$. This relation orders $X$ (being transitive, antisymmetric, and strictly irreflexive), and it is clear that a point of $X$ is a $d$-point for $h$ if and only if it is minimal with respect to $\ll$.

(iii) If $f$ is a function $X \to X$, it may be possible to choose $d$ and $h$ so that the relation $\ll$ has the property that if $f(x) \neq x$ then $f(x) \ll x$ (or, more generally, $y \ll x$ for some $y$), and then any $d$-point for $h$ is a fixed point for $f$. Thus the first part of the above theorem yields the class of fixed-point theorems considered in [1]. In special cases, $h$ can be appropriately defined in terms of $d$ and $f$ by the formula $h(x) = \beta d(x, f(x))$, with suitable values for the constant $\beta$. For example, if $d$ and $f$ satisfy Banach's condition

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y$, where $\alpha < 1$, we can take $\beta = (1 - \alpha)^{-1}$; and if (as in [2])

$$d(f(x), f(y)) \leq \alpha(d(x, f(y)) + d(y, f(x)))$$

for all $x, y$, where $\alpha < \frac{1}{2}$, we can take $\beta = (1 - 2\alpha)^{-1}(1 - \alpha)$. (In each of these cases, it can be assumed that $X$ is the closure of the $f$-orbit of some point.)

**References**


**Department of Pure Mathematics, University College, Swansea SA2 8PP, Wales**