A CHARACTERIZATION OF METRIC COMPLETENESS

J. D. WESTON

Abstract. A proof is given of a theorem, relevant to fixed-point theory, which implies that a metric space \((X, d)\) is complete if and only if, for each continuous function \(h: X \to \mathbb{R}\) bounded below on \(X\), there is a point \(x_0\) such that \(h(x_0) - h(x) < d(x_0, x)\) for every other point \(x\).

If \((X, d)\) is a metric space and \(h\) is a function \(X \to \mathbb{R}\), by a \textit{d-point} for \(h\) we mean a point \(x_0\) of \(X\) such that, for every other point \(x\),

\[ h(x_0) - h(x) < d(x_0, x). \]

In terms of this notion, the following theorem gives a necessary and sufficient condition for \((X, d)\) to be complete. The necessity of the condition generalizes the proposition mentioned by Chi Song Wong in his recent note [1] concerning sufficient conditions for the existence of a fixed point for a function \(X \to X\). (For some comparable results, and other methods of proof, see the papers listed in [1], especially those of Brønsted and Ekeland.)

**Theorem.** If the metric space \((X, d)\) is complete then any lower semicontinuous function \(X \to \mathbb{R}\) which is bounded below has a \textit{d-point}. If \((X, d)\) is not complete there is a uniformly continuous function \(X \to \mathbb{R}\) which is bounded below but has no \textit{d-point}.

**Proof.** Suppose first that \((X, d)\) is complete, and let \(h\) be a function \(X \to \mathbb{R}\) which is lower semicontinuous (with respect to \(d\)) and is bounded below. Taking \(x_1\) to be any point of \(X\), we choose a sequence \(\{x_n\}\) in the following way. For each \(n\), let

\[ c_n = \inf \{ h(x) : h(x_n) - h(x) > d(x_n, x) > 0 \}, \]

Received by the editors November 19, 1976.


Key words and phrases. Completeness, fixed point, metric space, order, semicontinuity.
and let \( x_{n+1} \) be a point such that

\[
(1) \quad h(x_n) - h(x_{n+1}) \geq d(x_n, x_{n+1})
\]

and

\[
(2) \quad h(x_{n+1}) < c_n + n^{-1}.
\]

(If \( x_n \) is a \( d \)-point for \( h \) then \( x_{n+1} \) must be \( x_n \) and \( c_n = \infty \).) From (1) it follows that the sequence \( \{h(x_n)\} \) is nonincreasing, and that if \( m > n \) then

\[
(3) \quad h(x_n) - h(x_m) \geq d(x_n, x_m).
\]

Since the sequence \( \{h(x_n)\} \) is bounded below, it is convergent. Hence, by (3) and the assumed completeness, the sequence \( \{x_n\} \) is convergent: let \( x_0 \) be its limit. Now

\[
(4) \quad h(x_n) - h(x_0) \geq d(x_n, x_0)
\]

for every \( n \), because if, for some \( n \),

\[
h(x_n) - h(x_0) < d(x_n, x_0) - \epsilon,
\]

where \( \epsilon > 0 \), then by the lower semicontinuity of \( h \) there would be a neighbourhood \( U \) of \( x_0 \) such that

\[
h(x_n) - h(x) < d(x_n, x_0) - \epsilon
\]

for every \( x \) in \( U \), and then \( m \) could be such that \( x_m \in U \) and \( d(x_m, x_0) < \epsilon \), so that, contrary to (3),

\[
h(x_n) - h(x_m) < d(x_n, x_0) - \epsilon < d(x_n, x_m).
\]

If \( x_0 \) is not a \( d \)-point for \( h \) then, for some \( x \),

\[
(5) \quad h(x_0) - h(x) \geq d(x_0, x) > 0.
\]

From (4) (with \( n + 1 \) in place of \( n \)) and (2),

\[
h(x) < h(x_{n+1}) + h(x) - h(x_0) < c_n + n^{-1} + h(x) - h(x_0).
\]

Hence, by (5), we can choose \( n \) so that \( h(x) < c_n \). From (4) and (5),

\[
h(x) \geq h(x_n) > h(x),
\]

so that \( x_n \neq x \) and therefore \( d(x_n, x) > 0 \), and, moreover,

\[
h(x_n) - h(x) \geq d(x_n, x).
\]

It now follows from the definition of \( c_n \) that \( h(x) \geq c_n \), and we have a contradiction. Thus \( x_0 \) is a \( d \)-point for \( h \).

Now suppose that \( (X, d) \) is not complete, and let \( \{x_n\} \) be a Cauchy sequence (with respect to \( d \)) which is not convergent. For any point \( x \) of \( X \), \( \{2d(x, x_n)\} \)
is a Cauchy sequence in $\mathbb{R}$: let $h(x)$ be its limit. Then $h(x) > 0$, so the function $h$ is bounded below. Also, if $x_0 \in X$,

$$|h(x_0) - h(x)| \leq 2d(x_0, x),$$

so $h$ is uniformly continuous; and

$$\frac{1}{2}(h(x_0) + h(x)) \geq d(x_0, x),$$

so

$$h(x_0) - h(x) \geq d(x_0, x) + \frac{1}{2}(h(x_0) - h(x)).$$

Now, by the definition of $h$, $h(x_m) \to 0$ as $m \to \infty$; therefore $3h(x) < h(x_0)$ if $x = x_m$ and $m$ is large. Thus $x_0$ is not a $d$-point for $h$.

**Remarks.** (i) When $X = \mathbb{R}$, and $d$ is the usual metric for $\mathbb{R}$, a function $A \to \mathbb{R}$ which is uniformly continuous but not bounded below may have, but need not have, a $d$-point.

(ii) When $d$ and $h$ are given, a relation $\ll$ can be defined on $X$ by the stipulation that $x \ll y$ if and only if $h(y) - h(x) \geq d(x, y) > 0$. This relation orders $X$ (being transitive, antisymmetric, and strictly irreflexive), and it is clear that a point of $X$ is a $d$-point for $h$ if and only if it is minimal with respect to $\ll$.

(iii) If $f$ is a function $X \to X$, it may be possible to choose $d$ and $h$ so that the relation $\ll$ has the property that if $f(x) \neq x$ then $f(x) \ll x$ (or, more generally, $y \ll x$ for some $y$), and then any $d$-point for $h$ is a fixed point for $f$.

Thus the first part of the above theorem yields the class of fixed-point theorems considered in [1]. In special cases, $h$ can be appropriately defined in terms of $d$ and $f$ by the formula $h(x) = \beta d(x, f(x))$, with suitable values for the constant $\beta$. For example, if $d$ and $f$ satisfy Banach’s condition

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \text{for all } x, y,$$

where $\alpha < 1$, we can take $\beta = (1 - \alpha)^{-1}$; and if (as in [2])

$$d(f(x), f(y)) \leq \alpha [d(x, f(y)) + d(y, f(x))] \quad \text{for all } x, y,$$

where $\alpha < \frac{1}{2}$, we can take $\beta = (1 - 2\alpha)^{-1}(1 - \alpha)$. (In each of these cases, it can be assumed that $X$ is the closure of the $f$-orbit of some point.)

**References**
