PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 64, Number 1, May 1977

THE L^p MODULUS OF CONTINUITY AND FOURIER SERIES OF LIPSCHITZ FUNCTIONS

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ABSTRACT. This paper deals with certain inequalities about the L^p modulus of continuity, and some properties of the Fourier coefficients of functions in the Lipschitz spaces Λ_σ^{pq}. The inequalities are similar to those recently obtained by Garsia.

1. The L^p-modulus of continuity of $f \in L^p(\mathbf{R}^n)$ is defined by

$$\omega_p(f; h) = \sup_{|t| \leqslant h} \left\{ \int_{\mathbf{R}^n} |f(x+t) - f(x)|^p \, dx \right\}^{1/p}.$$

If f: $\mathbf{R}^n \to \mathbf{R}$ is periodic, its period being the torus $T^n = \{x = (x_1, \ldots, x_n)$: $0 < x_i \leqslant 2\pi\}$, then we write $f \in L^p(T^n)$, and $\omega_p(f; h)$ is defined as above except the integration is over T^n.

The first part of the paper consists of results that are similar to and extend some inequalities of Garsia [2], and in the second part we shall establish a general theorem about Fourier coefficients of functions satisfying

$$\|f\|_{\sigma; p,q} \equiv \|f\|_p + \left\{ \int_0^\pi \frac{\omega_p^q(f; t)}{t^{1+q\sigma}} \, dt \right\}^{1/q} < \infty, \tag{1}$$

where $\sigma > 0$, $0 < q < \infty$, $1 \leqslant p$. If $q \geqslant 1$ and $0 < \sigma < 1$, the above is a norm for the familiar Lipschitz spaces Λ_σ^{pq} [5], [6]. For notational convenience we will retain the symbol Λ_σ^{pq} to denote the class of all f for which (1) is finite, even if $0 < q < 1$ and $\sigma \geqslant 1$.

These results can be viewed as a generalization to Λ_σ^{pq} of Szasz's theorem [7_{I}, p. 243] which says that $\Sigma|\hat{f}(\nu)|^\gamma < \infty$ if $f \in \Lambda_\sigma$ and $\gamma > 2/(1+2\sigma)$, $0 < \sigma \leqslant 1$. For $\gamma = 2/(1+2\sigma)$, the above series may diverge, and our results will, in particular, single out the spaces Λ_σ^{pq} for which convergence at the critical index $2/(1+2\sigma)$ occurs. Garsia's theorem [2] on absolute convergence also appears as a special case.

2. For $f \in L^p(\mathbf{R}^n)$ it was proven in [4] that for $q \geqslant p$, $\lambda > 0$,

$$\|f\|_q \leqslant c \int_0^\lambda \frac{\omega_p(f; t)}{t^{1+n(1/p-1/q)}} \, dt + \frac{\|f\|_p}{\lambda^{n(1/p-1/q)}}, \tag{2}$$

where c depends only upon n. Actually, (2) is the last displayed inequality in

Received by the editors May 20, 1976.
AMS (MOS) subject classifications (1970). Primary 42A16, 26A14, 26A16.
Key words and phrases. L^p modulus of continuity, Lipschitz spaces, Fourier series.

the proof of Theorem 1 in [4]. In the periodic case $f \in L^p(T^n)$, we have the same inequality with $0 < \lambda \leqslant 2\pi$. If in addition $\int_{T^n} f\,dx = 0$, then we obtain, $q \geqslant p$,

$$\|f\|_q \leqslant c\int_0^{2\pi} \frac{\omega_p(f;t)}{t^{1+n(1/p-1/q)}}\,dt. \tag{3}$$

This is not explicitly stated in [4], but the proof of (3) is the same as the proof of Theorem 1 of [4] with the choice $\lambda = 2\pi$. If we let $q \to \infty$ in (3) we obtain the n-dimensional version of one of Garsia's theorems [2, p. 87].

REMARKS. (i) Garsia uses for the periodic 1-dimensional case

$$\omega_p^*(f;h) = \left\{\int_0^{2\pi} |f(x+h) - f(x)|^p\,dx\right\}^{1/p},$$

which is $\leqslant \omega_p(f;h)$. It is easy to interchange these two moduli of continuity. In fact [2, p. 91]

$$\omega_p(f;h) \leqslant \frac{10}{h}\int_0^h \omega_p^*(f;t)\,dt,$$

so that

$$\int_0^{2\pi} \frac{\omega_p(f;h)}{h^{1+n\alpha}}\,dh \leqslant c\int_0^{2\pi}\int_0^h \frac{\omega_p^*(f;t)}{h^{2+n\alpha}}\,dt\,dh \leqslant c\int_0^{2\pi} \frac{\omega_p^*(f;t)}{t^{1+n\alpha}}\,dt,$$

by an application of Fubini's theorem.

(ii) The inequality (2) has recently been generalized by Blozinski [1] to a Lorentz space, $L(p,q)$, setting.

THEOREM 1. *If $f \in L^p(\mathbf{R}^n$ or $T^n)$ and $q \geqslant p$, then*

$$\omega_q(f;\tau) \leqslant c\int_0^\tau \frac{\omega_p(f;h)}{h^{1+n(1/p-1/q)}}\,dh,$$

where c depends only on n.

PROOF. For fixed t, let $\phi_t(x) = |f(x+t) - f(x)|$. Then

$$|\phi_t(x+h) - \phi_t(x)| \leqslant |f(x+h+t) - f(x+t)| + |f(x+h) - f(x)|,$$

and hence $\omega_p(\phi_t;h) \leqslant 2\omega_p(f;h)$. In (2) we replace f by ϕ_t, and obtain

$$\|\phi_t\|_q \leqslant c\left\{\int_0^\lambda \frac{\omega_p(f;\tau)}{\tau^{1+n(1/p-1/q)}}\,d\tau + \frac{\omega_p(f;t)}{\lambda^{n(1/p-1/q)}}\right\}.$$

We let now $\lambda = h$, $|t| \leqslant h$, and obtain

$$\omega_q(f;h) \leqslant c\left\{\int_0^h \frac{\omega_p(f;\tau)}{\tau^{1+n(1/p-1/q)}}\,d\tau + \frac{\omega_p(f;h)}{h^{n(1/p-1/q)}}\right\}.$$

Next we observe that for $\alpha = n(1/p - 1/q)$,

$$\int_h^{2h} \frac{\omega_p(f;\tau)}{\tau^{1+\alpha}}\,d\tau \geqslant c\,\frac{\omega_p(f;h)}{h^\alpha},$$

and consequently

$$\frac{\omega_p(f;h)}{h^\alpha} \leqslant c\int_0^h \frac{\omega_p(f;\tau)}{\tau^{1+\alpha}}\,d\tau.$$

REMARKS. (i) Garsia's theorem [2, p. 85], [3, p. 168] is obtained by letting $q \to \infty$.

(ii) Let T_j: $L^p \to L^p$ be a sequence of linear operators such that $\|T_jf\|_p \leqslant A_p\|f\|_p$ for each j (T_j could be a truncated singular integral operator [5]). If T_jf is continuous, it follows from Theorem 1 that

$$|T_jf(x) - T_jf(y)| \leqslant A_p\int_0^{|x-y|} \frac{\omega_p(f;\tau)}{\tau^{1+n/p}}\,d\tau,$$

and hence $\{T_jf\}$ is equicontinuous for all $f \in \Lambda^{p_1}_{n/p}$. Garsia's theorem [2, p. 86] is concerned with the case $T_jf = j$th partial sum of the Fourier series of f.

3. For the remainder of the paper we will restrict ourselves to $n = 1$ and the 2π-periodic case. For $f \in L^1$ let $f \sim \sum c_\nu e^{i\nu x}$ be the Fourier expansion of f. It will be useful to consider fractional differences of f, i.e., for $\gamma > 0$, let

$$\Delta_h^\gamma f(x) = \sum_{j=0}^\infty (-1)^j\binom{\gamma}{j} f(x - jh).$$

It is immediately checked that

$$\Delta_h^\gamma f(x) \sim \sum (1 - e^{-i\nu h})^\gamma c_\nu e^{i\nu x}.$$

LEMMA 1. *If* $1 \leqslant q < \infty$, *let* $0 < \sigma < \gamma$, *and if* $0 < q < 1$, *let* $\gamma > 1/q + \sigma - 1$. *Then*

$$\left\{\int_0^\pi \frac{\|\Delta_h^\gamma f\|_p^q}{h^{1+q\sigma}}\,dh\right\}^{1/q} \leqslant A\|f\|_{\sigma;p,q}.$$

PROOF. We consider first the case $1 \leqslant q < \infty$ and hence $\gamma > \sigma$. If $C = \|\Delta_h^\gamma f\|_p/h^{1/q+\sigma}$, then, since

$$\Delta_h^\gamma f(x) = \sum_{j=0}^\infty (-1)^j\binom{\gamma}{j}[f(x - jh) - f(x)],$$

$$C \leqslant \sum \left|\binom{\gamma}{j}\right| \frac{\omega_p(f;jh)}{h^{1/q+\sigma}}.$$

Hence

$$\|C\|_q \leqslant \sum \left|\binom{\gamma}{j}\right| \left\{\int_0^\pi \frac{\omega_p^q(f;jh)}{h^{1+q\sigma}}\,dh\right\}^{1/q}$$

$$= \sum \left|\binom{\gamma}{j}\right| j^\sigma \left\{\int_0^{j\pi} \frac{\omega_p^q(f;\tau)}{\tau^{1+q\sigma}}\,d\tau\right\}^{1/q}.$$

We split $\int_0^{j\pi} \leqslant \int_0^\pi + \int_\pi^\infty$, and since $\omega_p(f;\tau) \leqslant 2\|f\|_p$, $\int_\pi^\infty \leqslant c\|f\|_p^q$. Finally, since $\gamma > \sigma$ and $\binom{\gamma}{j} = O(j^{-1-\gamma})$, $\sum|\binom{\gamma}{j}|j^\sigma < \infty$, and the proof is complete.

The case $0 < q < 1$ and $\gamma > 1/q + \sigma - 1$ leads to the inequality

$$\int_0^\pi \frac{\|\Delta_h^\gamma f\|_p^q}{h^{1+q\sigma}}\, dh \leqslant \sum \left|\binom{\gamma}{j}\right|^q \int_0^\pi \frac{\omega_p^q(f; jh)}{h^{1+q\sigma}}\, dh$$

$$= \sum \left|\binom{\gamma}{j}\right|^q j^{q\sigma} \int_0^{j\pi} \frac{\omega_p^q(f;\tau)}{\tau^{1+q\sigma}}\, d\tau.$$

The choice of γ guarantees the convergence of $\sum|\binom{\gamma}{j}|^q j^{q\sigma}$, and estimating $\int_0^{j\pi}$ as before, the proof is complete.

THEOREM 2. *Let* $1 \leqslant p \leqslant 2$, $0 < q \leqslant p'$ (*if* $1 < p \leqslant 2$) $0 < q < \infty$ (*if* $p = 1$), *and* $\sigma > 0$. *Then*

$$\left\{ \sum_{\nu \neq 0} |\nu|^{q(1/p'+\sigma)-1} |c_\nu|^q \right\}^{1/q} \leqslant A \|f\|_{\sigma; p,q},$$

where A depends only on p, q, σ.

PROOF. If $1 \leqslant q \leqslant p'$, choose $\sigma < \gamma < \sigma + 1/q$, and if $0 < q < 1$, let $1/q + \sigma - 1 < \gamma < \sigma + 1/q$.

We will first consider the case $1 < p \leqslant 2$. By the Hausdorff-Young inequality [7_{II}, p. 101],

$$\left\{ \sum_{\nu \neq 0} |c_\nu|^{p'} \left|(1 - e^{-i\nu h})^\gamma\right|^{p'} \right\}^{1/p'} \leqslant A \|\Delta_h^\gamma f\|_p.$$

Since $|(1 - e^{-i\nu h})| \geqslant 2|\nu h|/\pi$ if $|\nu h| \leqslant \pi/2$, we obtain that

$$\left\{ \sum_{0<|\nu|<\pi/2h} |\nu|^{\gamma p'} |c_\nu|^{p'} \right\}^{1/p'} \leqslant A \frac{\|\Delta_h^\gamma f\|_p}{h^\gamma}.$$

We wish to estimate $\sum_{0<|\nu|<\pi/2h} |\nu|^{q(1/p'+\sigma)} |c_\nu|^q$, and to this end we write $|\nu| = |\nu|^\beta |\nu|^{1-\beta}$, where $\beta(1/p' + \sigma) = \gamma$. Since $\gamma < \sigma + 1/q$, an easy calculation gives

$$(1 - \beta) q (1/p' + \sigma) p' / (p' - q) > -1.$$

By Hölder's inequality

$$\sum_{0<|\nu|<\pi/2h} |\nu|^{q(1/p'+\sigma)} |c_\nu|^q \leqslant \left\{ \sum |\nu|^{\gamma p'} |c_\nu|^{p'} \right\}^{q/p'}$$

$$\cdot \left\{ \sum |\nu|^{(1-\beta)(1/p'+\sigma)p'/(p'-q)} \right\}^{(p'-q)/p'}$$

where the last two summations extend over $0 < |\nu| < \pi/2h$. The last term $\{\ldots\}^{(p'-q)/p'} \leqslant A \cdot (1/h)^{1-\gamma q+\sigma q}$. Hence

$$\sum_{0<|\nu|<\pi/2h} |\nu|^{q(1/p'+\sigma)} |c_\nu|^q \leqslant A \frac{\|\Delta_h^\gamma f\|_p^q}{h^{1+q\sigma}}.$$

If we integrate this inequality with respect to h from 0 to π, and apply Fubini's theorem on the left side, we get

$$\left\{ \sum_{\nu \neq 0} |\nu|^{q(1/p'+\sigma)-1} |c_\nu|^q \right\}^{1/q} \leqslant A \left\{ \int_0^\pi \frac{\|\Delta_h^\gamma f\|_p^q}{h^{1+q\sigma}}\, dh \right\}^{1/q}.$$

An application of Lemma 1 completes the proof for the case $1 < p \leqslant 2$.

If $p = 1$, then with the same choice of γ as before,

$$\sum_{0<|\nu|<\pi/2h} |\nu|^{q\sigma} |c_\nu|^q = \sum_{0<|\nu|<\pi/2h} |\nu|^{q\gamma} |c_\nu|^q |\nu|^{q(\sigma-\gamma)}$$

$$\leqslant A \frac{\|\Delta_h^\gamma f\|_1^q}{h^{q\gamma}} \sum_{0<|\nu|<\pi/2h} |\nu|^{q(\sigma-\gamma)}.$$

Since $q(\sigma - \gamma) > -1$, the last sum is majorized by $(1/h)^{q\sigma - q\gamma + 1}$ from which

$$\sum_{0<|\nu|\leqslant\pi/2h} |\nu|^{q\sigma} |c_\nu|^q \leqslant A \frac{\|\Delta_h^\gamma f\|_1^q}{h^{1+q\sigma}},$$

and the proof can be completed as before.

The case $q > p'$ requires a different exponent for $|\nu|$ as seen in the next theorem.

THEOREM 3. *Let* $1 < p \leqslant 2, p' < q < \infty$, *and* $\sigma > 0$. *Then*

$$\left\{ \sum |\nu|^{q\sigma} |c_\nu|^q \right\}^{1/q} \leqslant A \|f\|_{\sigma;\, p,q},$$

where A does not depend upon f.

PROOF. We choose $\sigma < \gamma < \sigma + 1/q$. We proceed as in the proof for Theorem 2, and since $p' < q$,

$$\left\{ \sum_{|\nu|<\pi/2h} |\nu|^{\gamma q} |c_\nu|^q \right\}^{1/q} \leqslant \left\{ \sum_{|\nu|<\pi/2h} |\nu|^{\gamma p'} |c_\nu|^{p'} \right\}^{1/p'} \leqslant A \frac{\|\Delta_h^\gamma f\|_p}{h^\gamma}.$$

Again we wish to estimate $\sum_{|\nu|<\pi/2h} |\nu|^{q\sigma+1} |c_\nu|^q$, and for that purpose we write $|\nu|^{q\sigma+1} = |\nu|^{\gamma q} |\nu|^{q\sigma+1-\gamma q}$. Then

$$\sum_{|\nu|<\pi/2h} |\nu|^{q\sigma+1} |c_\nu|^q \leqslant A \frac{\|\Delta_h^\gamma f\|_p^q}{h^{\gamma q}} \left(\frac{1}{h}\right)^{q\sigma-\gamma q+1} = A \frac{\|\Delta_h^\gamma f\|_p^q}{h^{1+q\sigma}}.$$

As before we integrate with respect to h and apply Lemma 1.

4. In this section we compare the above theorems with known results, and we will give examples showing that the exponents of $|\nu|$, i.e., $q(1/p' + \sigma) - 1$ if $0 < q \leqslant p'$, and $q\sigma$ if $p' < q < \infty$, cannot be interchanged.

(i) Garsia's result on absolute convergence [2, p. 88] is Theorem 2 with $q = 1$, $\sigma = 1/p$. Also note that Theorem 2 includes $p = 1$, a case not treated in [2].

(ii) If we let $p = p' = 2$ and $q = 2/(1 + 2\sigma)$, then Theorem 2 gives

$$\left\{ \sum |c_\nu|^q \right\}^{1/q} \leqslant A \|f\|_{\sigma;\, 2,q}.$$

It is of interest to recall here Szasz's theorem [7_{I}, p. 243], which says if

$f \in \Lambda_\sigma$, $0 < \sigma \leqslant 1$, and $\beta > 2/(1 + 2\sigma)$, then $\sum |c_\nu|^\beta < \infty$, but not necessarily for $\beta = 2/(1 + 2\sigma)$. The above inequality singles out the Lipschitz space for which convergence at the critical index occurs.

(iii) Under the hypothesis of Theorem 2, we show now that the left side cannot be improved to $\sum |\nu|^{q\sigma} |c_\nu|^q < \infty$ as Theorem 3 may lead one to expect. We let

$$f(x) = \sum \frac{e^{in \log n}}{n^{1/2+\sigma}} e^{inx} \in \Lambda_\sigma$$

[7_{I}, p. 197]. For $\beta = \sigma/2$, it is easily checked that $f \in \Lambda_\beta^{21}$, but $\sum |\nu|^{\beta-\alpha-1/2} = \infty$.

(iv) The left side of Theorem 3 cannot be improved to

$$L = \sum |\nu|^{q(1/p'+\sigma)-1} |c_\nu|^q < \infty.$$

As an example we consider

$$f(x) = \sum_1^\infty \frac{1}{2^{\nu\sigma} \cdot \nu^\delta} e^{i2^\nu x}$$

which is in Λ_σ^{pq} if $1 \leqslant p$, $1 \leqslant q$, $0 < \sigma < 1$, $\delta \geqslant 0$, and $q\delta > 1$ [6, p. 471]. If $q > p'$, then $L = \infty$.

(v) If $f \in \Lambda_\sigma$, then $c_\nu = O(\nu^{-\sigma})$, and O cannot be replaced by o. However, if $f \in \Lambda_\sigma^{pq}$, $1 \leqslant q < \infty$, then $|c_\nu| = o(\nu^{-\sigma})$, an immediate consequence of Theorems 1, 2.

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