PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 64, Number 1, May 1977

SET OF UNIQUENESS ON NONCOMMUTATIVE LOCALLY COMPACT GROUPS

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ABSTRACT. Using the terminology of P. Eymard we adapt the notion of set of uniqueness to noncommutative case and we show that every compact and residual set in a locally compact nondiscrete group is a set of uniqueness.

In this note we shall prove that every compact and residual (i.e. including no nonempty perfect set) set in locally compact nondiscrete group G is a set of uniqueness. In the particular case, every countable compact set is a set of uniqueness. For the case when the group G is a torus, this is a classical result of W. H. Young (see [1], [3], [8]). For commutative groups that fact was proved by L. H. Loomis [4].

We refer to P. Eymard [2] for the basic definitions, properties and theorems of $A(G)$, $B(G)$, $VN(G)$ and $C_\rho^*(G)$.

Now we recall some facts. $C_\rho^*(G)$ and $VN(G)$ are, respectively, the C^*-algebra and the von Neumann algebra generated by the operators $\rho(f)$ on $L^2(G)$, where ρ is the left regular representation of the group G and f is an arbitrary function on G with the compact support.

As in [2], we denote $C^*(G)$, the enveloping C^*-algebra of $L^1(G)$, i.e. the completion of the algebra $L^1(G)$ with respect to the norm

$$\|f\|_{C^*(G)} = \sup\{\|\pi(f)\| : \pi \in \Sigma\},$$

where Σ denotes the space of all $*$-representations of $L^1(G)$ on a Hilbert space.

The Fourier-Stieltjes algebra $B(G)$ consists of all finite complex linear combinations of continuous positive definite functions on G. As shown in [2], $B(G)$, as the dual of $C^*(G)$, is a commutative Banach algebra with unit (the multiplication is defined pointwise). The algebra $A(G)$ is defined as the norm closure in $B(G)$ of the functions from $B(G)$ with compact supports. We note also that $VN(G)$ is the dual of $A(G)$.

We recall also from [2] that if $T \in VN(G)$ and $u \in B(G)$, we can define the multiplication $u \cdot T \in VN(G)$ in the following way: for every function v from $A(G)$, $u \cdot T$ is a functional on $A(G)$ such that $\langle v, u \cdot T\rangle = \langle uv, T\rangle$.

Received by the editors January 14, 1976.
AMS (MOS) subject classifications (1970). Primary 43A55, 22D25.
Key words and phrases. Set of uniqueness, $C_\rho^*(G)$-algebra.

As in [2], the element a from G belongs to the support of the operator $T \in VN(G)$ ($a \in \operatorname{supp}(T)$), if the following equivalent statements hold:

(i) For every neighborhood V of a, there exists a function $u \in A(G)$ with support in V such that $\langle u, T \rangle \neq 0$.

(ii) If $u \in A(G)$ and $u \cdot T = 0$, then $u(a) = 0$.

Now we can introduce in general case the following

DEFINITION 1. Let E be a compact subset of G; a set E is called *a set of uniqueness* if $T \in C_\rho^*(G)$ and $\operatorname{supp}(T) \subseteq E$ imply $T = 0$.

We note that if the group G is commutative and $\hat{G}$ is the dual group, then $C_\rho^*(G)$ and $VN(G)$, respectively, coincide with the spaces $C_0(\hat{G})$ and $L^\infty(\hat{G})$, and the support of $T \in VN(G)$ coincides with the spectrum of its inverse Fourier transform which is an element of $L^\infty(\hat{G})$.

Thus, for the commutative groups, Definition 1 reduces to the usual notion of a set of uniqueness (see e.g. Y. Meyer [5]).

PROPOSITION 1. *If the group G is nondiscrete, then $C_\rho^*(G) \cap C_\rho^*(G_d) = \{0\}$, where G_d is the group G with discrete topology.*

PROOF. We recall that every $S \in C_\rho^*(G_d)$ is of the form $S = \rho(F)$, $F = S(\delta_e)$, $\|S\|_\rho \geqslant \|F\|_2 =^{\mathrm{df}} \|S\|_2$, and $S = 0$ if and only if $\|S\|_2 = 0$. Now we show that

$$\operatorname{dist}(S, C_\rho^*(G)) \geqslant \|S\|_2 \quad \text{for} \quad S \in C_\rho^*(G_d). \tag{1}$$

Using the density argument it suffices to prove that

$$\|T - \rho(f)\|_\rho \geqslant \|T\|_2 \tag{2}$$

for every continuous function f on G with compact support T of the form $T = \sum_{n=1}^N a_n \rho(x_n)$. Let $\{V_\alpha\}$ be a family of symmetric neighborhoods of identity such that the Haar measure of V_α tends to zero, and put $h_\alpha = |V|^{-1/2}\chi_{v_\alpha}$, where χ_{v_α} is the characteristic function of the set V_α. Since the net $u_\alpha = |V_\alpha|^{-1/2} h_\alpha$ is an approximate unit in $L^2(G)$, we obtain

$$\lim_{|V_\alpha| \to 0} \|f * h_\alpha\|_2 = 0. \tag{3}$$

Note also that

$$\lim_{|V_\alpha| \to 0} \|T(h_\alpha)\|_2 = \|T\|_2, \tag{4}$$

and that fact gives (2).

COROLLARY 1. *The algebra $C_\rho^*(G)$ has a unit if and only if the group G is discrete.*

Now we introduce the following auxiliary definition (see also [4]).

DEFINITION 2. Let $T \in VN(G)$ and $x \in G$. We say that T belongs locally to $C_\rho^*(G_d)$ at the point x ($T \in C_\rho^*(x)$) if there exists a function $u \in A(G)$ such that $U(x) \neq 0$ and $u \cdot T \in C_\rho^*(G_d)$.

REMARK. Since every function from $A(G)$ is continuous, so for every T from $VN(G)$ the set $A_T = \{x \in G\colon T \in C^*_\rho(x)\}$ is open.

LEMMA 1. *Let $T \in VN(G)$ and the support of T be compact. Then*:

(a) *if u_1 and u_2 from $B(G)$ are identical on a neighborhood of the support of T, then $u_1 T = u_2 T$.*

(b) *$T \in C^*_\rho(G_d)$ if and only if, for every $x \in G$, we have $T \in C^*_\rho(x)$.*

The proof is the same as in Abelian case (see [2], [4]).

LEMMA 2. *$C^*_\rho(G)$ and $C^*_\rho(G_d)$ are $B(G)$-modules.*

The proof follows from the following two facts (see [2]):

(1) If $T = \rho(\mu)$, where μ is a Borel regular measure on G and $v \in B(G)$, then $v \cdot \rho(\mu) = \rho \cdot (v\mu)$.

(2) If $T \in VN(G)$ and $u \in B(G)$, then $\|u \cdot T\|_\rho < \|u\|_B \cdot \|T\|_\rho$.

LEMMA 3. *If the support of $T \in VN(G)$ is compact, then $T \in C^*_\rho(x)$ for every x which is not in the support of T.*

PROOF. From the regularity of the algebra $A(G)$ there exists a function $u \in A(G)$ such that $u = 0$ on a neighborhood of the support of T and $u(x) \neq 0$. So by the Lemma 1(a), $u \cdot T = 0$; but this means that $T \in C^*_\rho(x)$.

We now can prove the following

PROPOSITION 2. *Let $T \in VN(G)$ and let T have a compact support. Then if $T \in C^*_\rho(x)$ for every $x \neq e$, then $u \cdot T \in C^*_\rho(G_d)$ for every $u \in B(G)$ such that $u(e) = 0$.*

PROOF. Since the support of T is compact, there exists a function $v \in A(G)$ such that $v = 1$ on the neighborhood of the support of T. Hence by Lemma 1(a), $v \cdot T = T$. But the Fourier algebra $A(G)$ is an ideal in $B(G)$ so uv is in $A(G)$ and $uv(e) = 0$. Now we can approximate uv in $A(G)$ norm by a function $g \in A(G)$ which vanishes in a neighborhood of the identity. (See [2, Corollary (4.11)].) Because $\operatorname{supp}(g \cdot T) \subset \operatorname{supp}(g) \cap \operatorname{supp}(T)$, so the identity is not in the support of $g \cdot T$; hence, by Lemma 3, $g \cdot T \in C^*_\rho(e)$. On the other hand, it is obvious that $g \cdot T \in C^*_\rho(x)$ for every $x \neq e$, so by Lemma 1(b), $g \cdot T \in C^*_\rho(G_d)$. Since $u \cdot T = (uv) \cdot T$ and $C^*_\rho(G_d)$ is complete in the operator norm, Proposition 2 follows.

COROLLARY 2. *Let G be a locally compact nondiscrete group. If*

$$T \in C^*_\rho(G) \cap C^*_\rho(x) \quad \text{for every } x \neq e,$$

and T has a compact support, then $T = 0$.

PROOF. From Proposition 2 we have that $u \cdot T \in C^*_\rho(G_d)$ for every $u \in B(G)$ such that $u(e) = 0$. But also from Lemma 2, $u \cdot T \in C^*_\rho(G)$, so by Proposition 1 we have $u \cdot T = 0$. From condition (ii) of the definition of the support of the operator T, we obtain $u(a) = 0$ for every $u \in A(G)$ such that $u(e) = 0$ and every point $a \in \operatorname{supp}(T)$. But if $T \neq 0$, then we can consider two cases: either $\operatorname{supp}(T) = \{e\}$ or $\operatorname{supp}(T) \supset \{e, a\}$ for some $a \neq e$. In the first case $T = \lambda I$ (see [2, Theorem 4.9]). But $T \in C_\rho^*(G)$, so by Proposition 1 we get a contradiction; the second case is also impossible because we can take a function $u \in A(G)$ such that $u(a) \neq 0$ and $u(e) = 0$.

Now we are in position to prove

THEOREM. *If E is compact and residual in a nondiscrete group G, then E is a set of uniqueness.*

PROOF. Let T be in $C_\rho^*(G)$ with support in E. We know that the set A_T is open and A_T^c, the complement of A_T in G, is contained in the support of T. We shall prove that A_T^c is empty.

Let A_T^c be nonempty and let x_0 be an isolated point of A_T^c; we also can assume that $x_0 = e$. From the regularity of $A(G)$, there exists a function $u \in A(G)$ such that $u(e) \neq 0$ and $u(x) = 0$ for x from $A_T^c \setminus \{e\}$. Hence by Lemma 3, $u \cdot T \in C_\rho^*(x)$ for every $x \neq e$ and also $u \cdot T \in C_\rho^*(G)$. So by the Corollary 2 $u \cdot T = 0$, i.e. $e \not\in A_T^c$–a contradiction. Hence $A_T^c = \varnothing$ and, by Lemma 1(b), $T \in C_\rho^*(G_d)$. Finally, using Proposition 1, we obtain $T = 0$.

COROLLARY 3. *Every countable compact set in nondiscrete group is a set of uniqueness.*

REMARK. Using the same argument as in Abelian case one can show that a finite union of sets of uniqueness is also a set of uniqueness.

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