SET OF UNIQUENESS ON NONCOMMUTATIVE LOCALLY COMPACT GROUPS

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Abstract. Using the terminology of P. Eymard we adapt the notion of set of uniqueness to noncommutative case and we show that every compact and residual set in a locally compact nondiscrete group is a set of uniqueness.

In this note we shall prove that every compact and residual (i.e. including no nonempty perfect set) set in locally compact nondiscrete group $G$ is a set of uniqueness. In the particular case, every countable compact set is a set of uniqueness. For the case when the group $G$ is a torus, this is a classical result of W. H. Young (see [1], [3], [8]). For commutative groups that fact was proved by L. H. Loomis [4].

We refer to P. Eymard [2] for the basic definitions, properties and theorems of $A(G)$, $B(G)$, $VN(G)$ and $C^*_p(G)$.

Now we recall some facts. $C^*_p(G)$ and $VN(G)$ are, respectively, the $C^*$-algebra and the von Neumann algebra generated by the operators $\rho(f)$ on $L^2(G)$, where $\rho$ is the left regular representation of the group $G$ and $f$ is an arbitrary function on $G$ with the compact support.

As in [2], we denote $C^*(G)$, the enveloping $C^*$-algebra of $L^1(G)$, i.e. the completion of the algebra $L^1(G)$ with respect to the norm

$$\|f\|_{C^*(G)} = \sup\{\|\pi(f)\| : \pi \in \Sigma\},$$

where $\Sigma$ denotes the space of all $\ast$-representations of $L^1(G)$ on a Hilbert space.

The Fourier-Stieltjes algebra $B(G)$ consists of all finite complex linear combinations of continuous positive definite functions on $G$. As shown in [2], $B(G)$, as the dual of $C^*(G)$, is a commutative Banach algebra with unit (the multiplication is defined pointwise). The algebra $A(G)$ is defined as the norm closure in $B(G)$ of the functions from $B(G)$ with compact supports. We note also that $VN(G)$ is the dual of $A(G)$.

We recall also from [2] that if $T \in VN(G)$ and $u \in B(G)$, we can define the multiplication $u \cdot T \in VN(G)$ in the following way: for every function $v$ from $A(G)$, $u \cdot T$ is a functional on $A(G)$ such that $\langle v, u \cdot T \rangle = \langle uv, T \rangle$.

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As in [2], the element $a$ from $G$ belongs to the support of the operator $T \in \text{VN}(G)$ ($a \in \text{supp}(T)$), if the following equivalent statements hold:

(i) For every neighborhood $V$ of $a$, there exists a function $u \in A(G)$ with support in $V$ such that $\langle u, T \rangle \neq 0$.

(ii) If $u \in A(G)$ and $u \cdot T = 0$, then $u(a) = 0$.

Now we can introduce in general case the following

**Definition 1.** Let $E$ be a compact subset of $G$; a set $E$ is called a set of uniqueness if $T \in \mathcal{C}_\rho^*(G)$ and $\text{supp}(T) \subseteq E$ imply $T = 0$.

We note that if the group $G$ is commutative and $\hat{G}$ is the dual group, then $\mathcal{C}_\rho^*(G)$ and $\text{VN}(G)$, respectively, coincide with the spaces $C_0(\hat{G})$ and $L^\infty(\hat{G})$, and the support of $T \in \text{VN}(G)$ coincides with the spectrum of its inverse Fourier transform which is an element of $L^\infty(\hat{G})$.

Thus, for the commutative groups, Definition 1 reduces to the usual notion of a set of uniqueness (see e.g. Y. Meyer [5]).

**Proposition 1.** If the group $G$ is nondiscrete, then $\mathcal{C}_\rho^*(G) \cap \mathcal{C}_\rho^*(G_d) = \{0\}$, where $G_d$ is the group $G$ with discrete topology.

**Proof.** We recall that every $S \in \mathcal{C}_\rho^*(G_d)$ is of the form $S = \rho(F)$, $F = S(\delta_e)$, $\|S\|_\rho > \|F\|_2 = \|S\|_2$, and $S = 0$ if and only if $\|S\|_2 = 0$. Now we show that

$$\text{dist}(S, \mathcal{C}_\rho^*(G)) > \|S\|_2 \quad \text{for} \quad S \in \mathcal{C}_\rho^*(G_d).$$

Using the density argument it suffices to prove that

$$\|T - \rho(f)\|_\rho > \|T\|_2$$

for every continuous function $f$ on $G$ with compact support $T$ of the form $T = \sum_{n=1}^N a_n \rho(x_n)$. Let $\{V_n\}$ be a family of symmetric neighborhoods of identity such that the Haar measure of $V_n$ tends to zero, and put $h_a = |V_n|^{-1/2} x_n$, where $x_n$ is the characteristic function of the set $V_n$. Since the net $u_a = |V_n|^{-1/2} h_a$ is an approximate unit in $L^2(G)$, we obtain

$$\lim_{|V_n| \to 0} \|f \ast h_a\|_2 = 0.$$

Note also that

$$\lim_{|V_n| \to 0} \|T(h_a)\|_2 = \|T\|_2,$$

and that fact gives (2).

**Corollary 1.** The algebra $\mathcal{C}_\rho^*(G)$ has a unit if and only if the group $G$ is discrete.

Now we introduce the following auxiliary definition (see also [4]).

**Definition 2.** Let $T \in \text{VN}(G)$ and $x \in G$. We say that $T$ belongs locally to $\mathcal{C}_\rho^*(G)$ at the point $x$ ($T \in \mathcal{C}_\rho^*(x)$) if there exists a function $u \in A(G)$ such that $U(x) \neq 0$ and $u \cdot T \in \mathcal{C}_\rho^*(x)$. 

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Remark. Since every function from $A(G)$ is continuous, so for every $T$ from $VN(G)$ the set $A_T = \{ x \in G : T \in C_p^*(x) \}$ is open.

Lemma 1. Let $T \in VN(G)$ and the support of $T$ be compact. Then:
(a) if $u_1$ and $u_2$ from $B(G)$ are identical on a neighborhood of the support of $T$, then $u_1T = u_2T$.
(b) $T \in C_p^*(G_d)$ if and only if, for every $x \in G$, we have $T \in C_p^*(x)$.

The proof is the same as in Abelian case (see [2], [4]).

Lemma 2. $C_p^*(G)$ and $C_p^*(G_d)$ are $B(G)$-modules.

The proof follows from the following two facts (see [2]):
(1) If $T = \rho(\mu)$, where $\mu$ is a Borel regular measure on $G$ and $v \in B(G)$, then $v \cdot \rho(\mu) = \rho(v\mu)$.
(2) If $T \in VN(G)$ and $u \in B(G)$, then $\| u \cdot T \|_p < \| u \|_B \cdot \| T \|_p$.

Lemma 3. If the support of $T \in VN(G)$ is compact, then $T \in C_p^*(x)$ for every $x$ which is not in the support of $T$.

Proof. From the regularity of the algebra $A(G)$ there exists a function $u \in A(G)$ such that $u = 0$ on a neighborhood of the support of $T$ and $u(x) \neq 0$. So by the Lemma 1(a), $u \cdot T = 0$; but this means that $T \in C_p^*(x)$.

We now can prove the following

Proposition 2. Let $T \in VN(G)$ and let $T$ have a compact support. Then if $T \in C_p^*(x)$ for every $x \neq e$, then $u \cdot T \in C_p^*(G_d)$ for every $u \in B(G)$ such that $u(e) = 0$.

Proof. Since the support of $T$ is compact, there exists a function $v \in A(G)$ such that $v = 1$ on the neighborhood of the support of $T$. Hence by Lemma 1(a), $v \cdot T = T$. But the Fourier algebra $A(G)$ is an ideal in $B(G)$ so $uv$ is in $A(G)$ and $uv(e) = 0$. Now we can approximate $uv$ in $A(G)$ norm by a function $g \in A(G)$ which vanishes in a neighborhood of the identity. (See [2, Corollary (4.11)]). Because supp$(g \cdot T) \subset$ supp$(g) \cap$ supp$(T)$, so the identity is not in the support of $g \cdot T$; hence, by Lemma 3, $g \cdot T \in C_p^*(e)$. On the other hand, it is obvious that $g \cdot T \in C_p^*(x)$ for every $x \neq e$, so by Lemma 1(b), $g \cdot T \in C_p^*(G_d)$. Since $u \cdot T = (uv) \cdot T$ and $C_p^*(G_d)$ is complete in the operator norm, Proposition 2 follows.

Corollary 2. Let $G$ be a locally compact nondiscrete group. If

$T \in C_p^*(G) \cap C_p^*(x)$ for every $x \neq e$,

and $T$ has a compact support, then $T = 0$.

Proof. From Proposition 2 we have that $u \cdot T \in C_p^*(G_d)$ for every $u \in B(G)$ such that $u(e) = 0$. But also from Lemma 2, $u \cdot T \in C_p^*(G)$, so by Proposition 1 we have $u \cdot T = 0$. From condition (ii) of the definition of the
support of the operator $T$, we obtain $u(a) = 0$ for every $u \in A(G)$ such that $u(e) = 0$ and every point $a \in \text{supp}(T)$. But if $T \neq 0$, then we can consider two cases: either $\text{supp}(T) = \{e\}$ or $\text{supp}(T) \supset \{e, a\}$ for some $a \neq e$. In the first case $T = \lambda I$ (see [2, Theorem 4.9]). But $T \in C^*_p(G)$, so by Proposition 1 we get a contradiction; the second case is also impossible because we can take a function $u \in A(G)$ such that $u(a) \neq 0$ and $u(e) = 0$.

Now we are in position to prove

**Theorem.** If $E$ is compact and residual in a nondiscrete group $G$, then $E$ is a set of uniqueness.

**Proof.** Let $T$ be in $C^*_p(G)$ with support in $E$. We know that the set $A_T$ is open and $A_T^c$, the complement of $A_T$ in $G$, is contained in the support of $T$. We shall prove that $A_T^c$ is empty.

Let $A_T^c$ be nonempty and let $x_0$ be an isolated point of $A_T^c$; we also can assume that $x_0 = e$. From the regularity of $A(G)$, there exists a function $u \in A(G)$ such that $u(e) \neq 0$ and $u(x) = 0$ for $x$ from $A_T^c \setminus \{e\}$. Hence by Lemma 3, $u \cdot T \in C^*_p(x)$ for every $x \neq e$ and also $u \cdot T \in C^*_p(G)$. So by the Corollary 2 $u \cdot T = 0$, i.e. $e \not\in A_T^c$—a contradiction. Hence $A_T^c = \emptyset$ and, by Lemma 1(b), $T \in C^*_p(G_a)$. Finally, using Proposition 1, we obtain $T = 0$.

**Corollary 3.** Every countable compact set in nondiscrete group is a set of uniqueness.

**Remark.** Using the same argument as in Abelian case one can show that a finite union of sets of uniqueness is also a set of uniqueness.

**References**


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