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SHORTER NOTES

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HOMOLOGY TORI WITH NONTRIVIAL SERRE SPECTRAL SEQUENCE

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ABSTRACT. An example is constructed of a fibration over the torus which induces isomorphisms on all homology groups but which has a nontrivial Serre spectral sequence.

In [2] W. Dwyer showed that for abelian or finitely generated nilpotent groups π and finitely generated π-modules M, if $H_0(\pi, M) = 0$, then $H_i(\pi, M) = 0$ for all i. From this it readily follows by induction that if p: $E \to B$ is a fibration with connected fibre F such that H_*p is an isomorphism, if $\pi_1 B$ is either abelian or finitely generated nilpotent, and if H_*F is finitely generated over $\pi_1 B$, then the Serre spectral sequence of p is trivial ($H_i(B, H_j F) = 0$ for $j > 0$), even though $\tilde{H}_* F$ need not be trivial. In another important case [1], every fibration over the circle which induces isomorphisms on all homology groups again has trivial Serre spectral sequence. One might therefore conjecture that a fibration inducing isomorphisms on all homology groups had a trivial Serre spectral sequence, at least if the base space were nice enough. In this note we exhibit a counterexample with base space the torus $T = K(\mathbf{Z} \times \mathbf{Z}, 1)$.

Let $\pi = \mathbf{Z} \times \mathbf{Z}$, with generators σ and τ. Let M be the free abelian group on the symbols $x_{i,j}$ with i and j nonnegative integers. Let π act on M by $\sigma(x_{0,j}) = x_{0,j}$, $\sigma(x_{i,j}) = x_{i,j} - x_{i-1,j}$ if $i > 0$; $\tau(x_{i,0}) = x_{i,0}$, $\tau(x_{i,j}) = x_{i,j} - x_{i,j-1}$ if $j > 0$. Direct calculations show that $H_2(\pi, M) = \mathbf{Z}$ and $H_i(\pi, M) = 0$ for $i \neq 2$. (This example is due to Dwyer.)

Next fix $n \geqslant 2$. Form the fibration $p: X \to T$ with fibre $K(M, n)$ using the trivial (or another) twisted k-invariant. Examination of the Serre spectral sequence reveals that $H_i p$ is an isomorphism for $i < n + 2$. Let $X^{(n+1)}$ be the $(n + 1)$-skeleton of X and obtain E from $X^{(n+1)}$ by attaching an $(n + 2)$-cell to kill each (free) generator of $H_{n+1}X^{(n+1)}$. This is possible because the Hurewicz homomorphism in dimension $n + 1$ is surjective, as one can show

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by analyzing the homotopy fibre of $p|X^{(n+1)}$. Since $\pi_{n+1}T = 0$, $p|X^{(n+1)}$ extends to a map $f\colon E \to T$. Make f into a fibration; the fibre F is $(n-1)$-connected and $H_nF \cong \pi_nF \cong \pi_nE \cong \pi_nX \cong M$. In particular, $H_2(T, H_nF) = \mathbf{Z}$. On the other hand H_*f is an isomorphism.

References

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