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SPINOR GENERA OF UNIMODULAR Z-LATTICES IN QUADRATIC FIELDS

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ABSTRACT. Let L be a unimodular $\mathbf{Z}$-lattice on a quadratic space V over $\mathbf{Q}$, $\dim V \geqslant 3$, and let $\mathcal{O}$ be the ring of algebraic integers of the quadratic field $E = \mathbf{Q}(\sqrt{m})$. We explicitly calculate the number of proper spinor genera in the genus of the lattice $L \otimes_{\mathbf{Z}} \mathcal{O}$.

The genus of every indefinite unimodular integral quadratic form in three or more variables over the rational number field is known to contain exactly one isometry class. That is, in this particular setting, the local integral equivalence of forms is sufficient to imply their global integral equivalence, thus giving an analogue to the assertion for fractional equivalence guaranteed by the Hasse-Minkowski Theorem. However, if such a form is considered, by extending the ring of coefficients, as a form over the ring of algebraic integers of a quadratic extension of $\mathbf{Q}$, forms with coefficients from the overring may be integrally equivalent to the given form locally everywhere but not globally. The number of proper global isometry classes of such lattices is the proper class number of the form and it is our purpose here to compute these class numbers. It will be seen that this number is closely related to the number of rational primes dividing the discriminant of the quadratic extension.

In the case of a nonmodular $\mathbf{Z}$-lattice, the genus may contain many isometry classes. The lifting behavior of these classes in a quadratic extension of $\mathbf{Q}$ has been analyzed in **[EH3]** and for odd-degree extensions in **[EH2]**. While, in general, these classes may collapse in a quadratic extension, conditions are given in **[EH3]** which assure that the proper class number of the lifted lattice is no smaller than that of the lattice over $\mathbf{Z}$. While no explicit calculations are carried out here for nonmodular lattices, the considerations used will give a rough upper bound for the amount of growth of this class number in passing from $\mathbf{Q}$ to a quadratic extension of $\mathbf{Q}$.

We will adopt the geometric language of quadratic spaces and lattices; unexplained notations follow those of **[O]** and **[EH1]**. V will denote a quadratic space of dimension three or more over the rational number field $\mathbf{Q}$ and L a unimodular lattice on V. For any rational prime p, V_p and L_p denote

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the localizations $V \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $L \otimes_{\mathbf{Z}} \mathbf{Z}_p$, respectively. Let $E = \mathbf{Q}(\sqrt{m})$ be a quadratic extension of $\mathbf{Q}$ with ring of algebraic integers $\mathcal{O}$. The objects lifted to the field E will be denoted by $\tilde{V} = V \otimes_{\mathbf{Q}} E$ and $\tilde{L} = L \otimes_{\mathbf{Z}} \mathcal{O}$. Finally, if $\mathfrak{p}$ is a prime spot on E, we have $\tilde{V}_{\mathfrak{p}} = \tilde{V} \otimes_E E_{\mathfrak{p}}$ and $\tilde{L}_{\mathfrak{p}} = \tilde{L} \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}$, where $\mathcal{O}_{\mathfrak{p}}$ is the ring of integers at $\mathfrak{p}$.

The value which we will compute is the number $g^+(\tilde{L})$ of proper spinor genera in the genus of $\tilde{L}$. If $\tilde{L}$ is indefinite, this value is known to coincide with the number $h^+(\tilde{L})$ of proper isometry classes in the genus. By a result of Kneser (see [K]), this number can be computed by calculating the index of a certain subgroup of the idèle group of the field. Specifically,

$$g^+(\tilde{L}) = (J_E : P_{\tilde{D}} J_E^{\tilde{L}}),$$

where $\tilde{D} = \theta(O^+(\tilde{V}))$ and

$$J_E^{\tilde{L}} = \{ j \in J_E : j_{\mathfrak{p}} \in \theta(O^+(\tilde{L}_{\mathfrak{p}})) \text{ for all } \mathfrak{p} \in T \},$$

where T denotes the set of all nonarchimedean prime spots on E. Now L unimodular implies that $\theta(O^+(\tilde{L}_{\mathfrak{p}})) \supseteq \mathfrak{U}_{\mathfrak{p}} \dot{E}_{\mathfrak{p}}^2$ for all $\mathfrak{p} \in T$. So we can rewrite the above index as

$$(J_E : P_{\tilde{D}} J_E^{\tilde{L}}) = (J_E : P_E J_E^e)(P_E J_E^{\tilde{L}} : P_{\tilde{D}} J_E^{\tilde{L}})/(P_E J_E^{\tilde{L}} : P_E J_E^e)$$

where

$$J_E^e = \{ j \in J_E : \operatorname{ord}_{\mathfrak{p}}(j_{\mathfrak{p}}) \text{ is even for all } \mathfrak{p} \in T \}.$$

The first index can be readily identified. Let $\mathcal{I}(E)$ be the group of fractional ideals of E, $\mathcal{P}(E)$ the subgroup of principal ideals and $\mathcal{C}(E)$ the ideal class group of E, i.e., $\mathcal{C}(E) = \mathcal{I}(E)/\mathcal{P}(E)$.

1. Lemma. $J_E/P_E J_E^e \cong \mathcal{C}(E)/\mathcal{C}(E)^2$.

Proof. Consider the map

$$\Gamma : J_E \xrightarrow{\gamma} \mathcal{I}(E) \to \mathcal{I}(E)/\mathcal{P}(E) = \mathcal{C}(E) \to \mathcal{C}(E)/\mathcal{C}(E)^2$$

where γ is defined by the local conditions

$$|\gamma(j)|_{\mathfrak{p}} = |j_{\mathfrak{p}}|_{\mathfrak{p}} \quad \text{for all } \mathfrak{p} \in T,$$

and the remaining maps are the canonical ones. Γ is a well-defined surjective group homomorphism. Since every ideal $\mathfrak{A}$ with $\operatorname{ord}_{\mathfrak{p}}\mathfrak{A}$ even for all $\mathfrak{p} \in T$ is a square in $\mathcal{C}(E)$, $\ker(\Gamma)$ is precisely $P_E J_E^e$. Q.E.D.

For convenience of notation, we will refer to the indices $(P_E J_E^{\tilde{L}} : P_E J_E^e)$ and $(P_E J_E^{\tilde{L}} : P_{\tilde{D}} J_E^{\tilde{L}})$ as I_1 and I_2, respectively. We now calculate these indices explicitly.

In order for $I_1 \neq 1$, $\theta(O^+(L_{\mathfrak{p}}))$ must equal $\dot{E}_{\mathfrak{p}}$ for at least one dyadic spot $\mathfrak{p}$

on E. Thus, $\theta(O^+(L_2))$ must equal $\dot{\mathbf{Q}}_2$ by Proposition A of [H]. So suppose that this is the case. If 2 is ramified in E, it follows from Proposition A of [H] that $\theta(O^+(L_\mathfrak{p})) = \mathfrak{U}_\mathfrak{p}\dot{E}_\mathfrak{p}^2$ and $J_E^{\mathcal{L}} = J_E^e$. If 2 is inert in E, then $J_E^{\mathcal{L}} \neq J_E^e$. But if $j \in J_E^{\mathcal{L}} \backslash J_E^e$, $\text{ord}_{2\mathfrak{O}_E}(j_{2\mathfrak{O}_E})$ must be odd. Then $(2)j \in J_E^e$ and again $I_1 = 1$. So $I_1 \neq 1$ implies $m \equiv 1 \pmod 8$. The index in this case is determined by the following lemma.

2. Lemma. *Let* $m \equiv 1 \pmod 8$. *The following are equivalent when* $\theta(O^+(L_2)) = \dot{\mathbf{Q}}_2$:

(i) $I_1 \neq 1$.

(ii) $I_1 = 2$.

(iii) *There is no element* $\alpha \in E$ *with the property that* $\text{ord}_\mathfrak{Q}(\alpha)$ *is odd for one dyadic spot* $\mathfrak{Q}$ *and* $\text{ord}_\mathfrak{p}(\alpha)$ *is even for all* $\mathfrak{p} \in T$, $\mathfrak{p} \neq \mathfrak{Q}$.

(iv) *Neither* $+2$ *nor* -2 *is an element of* $N_{E/\mathbf{Q}}(\dot{E})$.

Proof. (i)$\Leftrightarrow$(ii). It suffices to show that $I_1 \leqslant 2$. Suppose that $\overline{(\alpha)j}$ and $\overline{(\beta)k}$ are two nontrivial elements of $P_E J_E^{\mathcal{L}} / P_E J_E^e$. Denote the spots lying over 2 by $\mathfrak{Q}$ and $\bar{\mathfrak{Q}}$. If both $\text{ord}_\mathfrak{Q}(j_\mathfrak{Q})$ and $\text{ord}_{\bar{\mathfrak{Q}}}(j_{\bar{\mathfrak{Q}}})$ are odd, then $(2)\, j \in J_E^e$; i.e., $j \in P_E J_E^e$. This is contrary to the choice of j. So suppose only $\text{ord}_\mathfrak{Q}(j_\mathfrak{Q})$ is odd. Then if $\text{ord}_\mathfrak{Q}(k_\mathfrak{Q})$ is odd while $\text{ord}_{\bar{\mathfrak{Q}}}(k_{\bar{\mathfrak{Q}}})$ is even, $jk \in J_E^e$. So $\overline{(\alpha)j} = \overline{(\beta)k}$. Finally, consider the case of $\text{ord}_\mathfrak{Q}(j_\mathfrak{Q})$ even and $\text{ord}_{\bar{\mathfrak{Q}}}(j_{\bar{\mathfrak{Q}}})$ odd. In this case $(2)\, jk \in J_E^e$ and again $\overline{(\alpha)j} = \overline{(\beta)k}$.

(i)$\Leftrightarrow$(iii). Suppose $I_1 = 1$. Let $j \in J_E^{\mathcal{L}}$ be defined by

$$j_\mathfrak{p} = \begin{cases} \pi & \text{if } \mathfrak{p} = \mathfrak{Q}, \\ 1 & \text{if } \mathfrak{p} \neq \mathfrak{Q}, \end{cases}$$

where $\mathfrak{Q} = \pi\mathfrak{O}_E$ is a dyadic spot on E. The condition $I_1 = 1$ implies the existence of $\alpha \in E$ with $(\alpha)j \in J_E^e$. For this α, $\text{ord}_\mathfrak{Q}(\alpha)$ is odd and $\text{ord}_\mathfrak{p}(\alpha)$ is even if $\mathfrak{p} = \mathfrak{Q}$. Conversely, suppose there exists $\alpha \in E$ with these order properties. Then for any $j \in J_E^{\mathcal{L}}$ either $(2)j$, $(2\alpha)j$ or $(\alpha)j$ is an element of J_E^e. In any case, $j \in P_E J_E^e$ and $I_1 = 1$.

(iii)$\Leftrightarrow$(iv) follows from Proposition 1.6 of [EH3]. Q.E.D.

The calculation of $I_2 = (P_E J_E^{\mathcal{L}} : P_{\tilde{D}} J_E^{\mathcal{L}})$ will be completed in two steps. Of course I_2 is trivially equal to one if $\tilde{D} = \dot{E}$. So we will here assume that $\tilde{D} \neq \dot{E}$. In this case, $(\dot{E} : \tilde{D}) = 4$ and the elements of $\tilde{D}$ are those positive elements of $\mathbf{Q}$ along with those elements of $\dot{E}\backslash\mathbf{Q}$ which are totally positive.

First we make the observation that $I_2 \leqslant 2$. Since the principal idèle $(-1) \in J_E^e \subseteq J_E^{\mathcal{L}}$, we have $(\alpha) \in P_{\tilde{D}} J_E^{\mathcal{L}}$ for any $\alpha \in \mathbf{Q}$. So let α and β be elements of $\dot{E}\backslash\mathbf{Q}$ with $N_{E/\mathbf{Q}}(\alpha) < 0$ and $N_{E/\mathbf{Q}}(\beta) < 0$. Then $N_{E/\mathbf{Q}}(\alpha\beta) > 0$ and $\alpha\beta \in \pm\tilde{D}$. So $(\alpha\beta) \in P_{\tilde{D}} J_E^{\mathcal{L}}$ and so $(\alpha) \equiv (\beta) \pmod{P_{\tilde{D}} J_E^{\mathcal{L}}}$.

3. Lemma. *Assume* $J_E^{\mathcal{L}} = J_E^e$ *and* $\tilde{D} \neq \dot{E}$. *Then*

$$I_2 = 1 \Leftrightarrow -1 \in N_{E/\mathbf{Q}}(\dot{E}).$$

Proof. ($\Rightarrow$) Let $\beta \in \dot{E}\backslash\pm\tilde{D}$. Since $I_2 = 1$, there exists some $\gamma \in \tilde{D}$ for which $(\beta) \in (\gamma)J_E^e$; that is, $(\beta\gamma) \in J_E^e$. It follows that $\operatorname{ord}_{\mathfrak{p}}(\beta) \equiv \operatorname{ord}_{\mathfrak{p}}(\gamma) \pmod 2$ for all $\mathfrak{p} \in T$ and so $\operatorname{ord}_p(N_{E/\mathbf{Q}}(\beta)) \equiv \operatorname{ord}_p(N_{E/\mathbf{Q}}(\gamma)) \pmod 2$ for all $p \in S$. Let $p_1, \ldots, p_s$ be the finite (possibly empty) set of rational primes for which $\operatorname{ord}_{p_i}(N_{E/\mathbf{Q}}(\beta)) \neq \operatorname{ord}_{p_i}(N_{E/\mathbf{Q}}(\gamma))$; say

$$\operatorname{ord}_{p_i}(N_{E/\mathbf{Q}}(\beta)) = 2n_{p_i} + \operatorname{ord}_{p_i}(N_{E/\mathbf{Q}}(\gamma))$$

for $n_{p_i} \in \mathbf{Z}$, for $i = 1, \ldots, s$. Let

$$\delta = \left(\prod_{i=1}^{s} p_i^{n_{p_i}}\right)\gamma.$$

Then

$$N_{E/\mathbf{Q}}(\delta) = \left(\prod_{i=1}^{s} p_i^{2n_{p_i}}\right)N_{E/\mathbf{Q}}(\gamma) = -N_{E/\mathbf{Q}}(\beta)$$

since $N_{E/\mathbf{Q}}(\beta) < 0$ and $\gamma \in \tilde{D}$ implies $N_{E/\mathbf{Q}}(\gamma) > 0$. Thus, $N_{E/\mathbf{Q}}(\delta\beta^{-1}) = -1$.

($\Leftarrow$) Let $(\beta) \in P_E J_E^{\mathcal{L}} \backslash P_{\tilde{D}} J_E^{\mathcal{L}}$. In particular, $\beta \in \dot{E}\backslash\mathbf{Q}$ and $N_{E/\mathbf{Q}}(\beta) < 0$. Now $-1 \in N_{E/\mathbf{Q}}(\dot{E})$, so there exists $\alpha \in \dot{E}$ with $N_{E/\mathbf{Q}}(\alpha) = -N_{E/\mathbf{Q}}(\beta)$. So $N_{E/\mathbf{Q}}(\alpha^{-1}\beta) = -1$ and $\operatorname{ord}_p(N_{E/\mathbf{Q}}(\alpha^{-1}\beta)) = 0$ for all rational primes p. So $\operatorname{ord}_{\mathfrak{p}}(\alpha^{-1}\beta) = 0$ for all prime spots $\mathfrak{p}$ on E which are either inert or ramified and $\operatorname{ord}_{\mathfrak{p}}(\alpha^{-1}\beta) = -\operatorname{ord}_{\bar{\mathfrak{p}}}(\alpha^{-1}\beta)$ where $\bar{\mathfrak{p}} \neq \mathfrak{p}$ is the conjugate of $\mathfrak{p}$. Let $p_1, \ldots, p_s$ be the set of split primes for which $\operatorname{ord}_{\mathfrak{p}}(\alpha^{-1}\beta)$ is odd. Then $\operatorname{ord}_{\mathfrak{p}}((\prod_{i=1}^{s} p_i)\alpha^{-1}\beta)$ is even for all spots $\mathfrak{p}$ on E. Writing $\gamma = (\prod_{i=1}^{s} p_i)\alpha^{-1}$, we have $(\beta) = (\gamma^{-1})(\gamma\beta)$ and $N_{E/\mathbf{Q}}(\gamma^{-1}) > 0$, $(\gamma\beta) \in J_E^e$. Q.E.D.

The case $J_E^{\mathcal{L}} \neq J_E^e$ remains; in particular, $m \equiv 1$ or $5 \pmod 8$.

4. Lemma. *Assume* $J_E^{\mathcal{L}} \neq J_E^e$ *and* $\tilde{D} \neq \dot{E}$.
(i) *For* $m \equiv 5 \pmod 8$, $I_2 = 1 \Leftrightarrow -1 \in N_{E/\mathbf{Q}}(\dot{E})$.
(ii) *For* $m \equiv 1 \pmod 8$, $I_2 = 1 \Leftrightarrow -1$ *or* $-2 \in N_{E/\mathbf{Q}}(\dot{E})$.

Proof. Suppose $I_2 = 1$. Let $\beta \in \dot{E}\backslash\pm\tilde{D}$; in particular, $N_{E/\mathbf{Q}}(\beta) < 0$. Then there exists $\gamma \in \dot{E}$ for which $\gamma \in \tilde{D}$ and $(\beta\gamma) \in J_E^{\mathcal{L}}$. If $(\beta\gamma) \in J_E^e$ it follows as in Lemma 3 that $-1 \in N_{E/\mathbf{Q}}(\dot{E})$. So suppose $\operatorname{ord}_{\mathfrak{p}}(\beta\gamma)$ is odd for some dyadic spot $\mathfrak{p}$. If $m \equiv 5 \pmod 8$ then $(2\beta\gamma) \in J_E^e$ with $N_{E/\mathbf{Q}}(2\beta\gamma) < 0$ and again $-1 \in N_{E/\mathbf{Q}}(\dot{E})$. So assume $m \equiv 1 \pmod 8$. If $\operatorname{ord}_{\mathfrak{p}}(\beta\gamma)$ is odd for both dyadic spots on E then again $(2\beta\gamma) \in J_E^e$ and $-1 \in N_{E/\mathbf{Q}}(\dot{E})$. If $\operatorname{ord}_{\mathfrak{p}}(\beta\gamma)$ is odd at precisely one of the dyadic spots on E, then $\operatorname{ord}_2(N_{E/\mathbf{Q}}(\beta\gamma))$ is odd and $\operatorname{ord}_{\mathfrak{p}}(N_{E/\mathbf{Q}}(\beta\gamma))$ is even for all odd primes p. The procedure used in Lemma 3 yields an element of norm -2.

To complete the proof of (i) it suffices to observe that the proof of the implication $-1 \in N_{E/\mathbf{Q}}(\dot{E})$ implies $I_2 = 1$ from Lemma 3 remains valid. To complete the converse in statement (ii), assume $-2 \in N_{E/\mathbf{Q}}(\dot{E})$. Then for any $\beta \in \dot{E}\backslash\pm D$ there exists $\alpha \in E$ for which $N_{E/\mathbf{Q}}(\alpha) = -2N_{E/\mathbf{Q}}(\beta) > 0$. So

$N_{E/\mathbf{Q}}(\alpha^{-1}\beta) = -\frac{1}{2}$ and there is a finite set $p_1, \ldots, p_s$ of odd primes for which $\mathrm{ord}_{\mathfrak{p}}((\prod_{i=1}^{s} p_i)\alpha^{-1}\beta)$ is even for all $\mathfrak{p} \nmid 2$. Write

$$(\beta) = \left(\left(\prod_{i=1}^{s} p_i^{-1}\right)\alpha\right)\left(\left(\prod_{i=1}^{s} p_i\right)\alpha^{-1}\beta\right) \in P_{\mathfrak{D}} J_E^{\mathcal{L}}. \quad \text{Q.E.D.}$$

5. SUMMARY. We are now in a position to calculate explicitly the values of $g^+(\mathcal{L})$, which we do in the following series of remarks.

(i) The conditions given in terms of the existence of elements of specified norm can be readily replaced by congruence conditions on the primes dividing m. The condition that α be a norm from $E = \mathbf{Q}(\sqrt{m})$ is equivalent to the representation of α by the quadratic space $[1, -m]$ over $\mathbf{Q}$, which is in turn equivalent to the local conditions that the Hilbert symbols $(\alpha, m)_p$ equal one for all rational primes p. Considering the cases pertinent to the preceding lemmas, $-1 \in N_{E/\mathbf{Q}}(\dot{E})$ if and only if $m > 0$, $m \equiv 1 \pmod 4$ when m odd (or $m/2 \equiv 1 \pmod 4$ when m even) and $p \equiv 1 \pmod 4$ for all odd primes p dividing m. Next, $2 \in N_{E/\mathbf{Q}}(\dot{E})$ if and only if $m \equiv \pm 1 \pmod 8$ when m odd (or $m/2 \equiv \pm 1 \pmod 8$ when m even) and $p \equiv \pm 1 \pmod 8$ for all odd primes p dividing m. Finally, $-2 \in N_{E/\mathbf{Q}}(\dot{E})$ if and only if $m > 0$, $m \equiv 1, 3 \pmod 8$ when m odd (or $m/2 \equiv -1, -3 \pmod 8$ when m even) and $p \equiv 1, 3 \pmod 8$ for all odd primes p dividing m.

(ii) From the above lemmas, it can be easily seen that $g^+(\mathcal{L})$ differs from $|\mathcal{C}(E)/\mathcal{C}(E)^2|$ by no more than a single factor of 2; that is

$$\tfrac{1}{2}|\mathcal{C}(E)/\mathcal{C}(E)^2| \leqslant g^+(\mathcal{L}) \leqslant 2|\mathcal{C}(E)/\mathcal{C}(E)^2|.$$

(iii) The value of the index $(J_E : P_E J_E^e)$ can be computed from known results on the structure of the ideal class group $\mathcal{C}(E)$ (see e.g. [HE]). In particular, if t denotes the number of rational primes which divide disc $(E/\mathbf{Q})$,

$$|\mathcal{C}(E)/\mathcal{C}(E)^2| = \begin{cases} 2^{t-1}, & \text{if } m < 0 \text{ or if } m > 0 \text{ with } -1 \in N_{E/\mathbf{Q}}(\dot{E}), \\ 2^{t-2}, & \text{if } m > 0 \text{ with } -1 \notin N_{E/\mathbf{Q}}(\dot{E}). \end{cases}$$

Of course, the norm condition here can be replaced by congruence conditions on the primes dividing $\mathrm{disc}(E/\mathbf{Q})$.

(iv) The exact value of $g^+(\mathcal{L})$ can be computed from Lemmas 1–4. If I denotes the quotient $I = I_2/I_1$, then $g^+(\mathcal{L}) = I \cdot |\mathcal{C}(E)/\mathcal{C}(E)^2|$. To distinguish the values of I, we follow the notation of [O] and call the unimodular lattice L "odd" if L represents an odd integer and "even" if L represents only even integers. The following table of values of I completes the calculation of $g^+(\mathcal{L})$ and is obtained from Lemmas 2–4 after eliminating several cases by the congruence considerations in (i).

	$m>0$			$m<0$		
	$\equiv 2, 3(4)$	$\equiv 1(8)$	$\equiv 5(8)$	$\equiv 2, 3(4)$	$\equiv 1(8)$	$\equiv 5(8)$
L indefinite L even	1	1	1	1	1	1
L odd	1	*	1	1	*	1
L definite L even	**	**	**	1	1	1
L odd	**	#	**	1	*	1

Values in the table are determined by:

$$(*)\qquad I=\begin{cases}1, & \text{if either } \pm 2\in N_{E/\mathbf{Q}}(\dot{E}),\\ \frac{1}{2}, & \text{otherwise;}\end{cases}$$

$$(**)\qquad I=\begin{cases}1, & \text{if } -1\in N_{E/\mathbf{Q}}(\dot{E}),\\ 2, & \text{otherwise;}\end{cases}$$

$$(\#)\qquad I=\begin{cases}2, & \text{if } 2\in N_{E/\mathbf{Q}}(\dot{E}) \text{ and } -1\notin N_{E/\mathbf{Q}}(\dot{E}),\\ \frac{1}{2}, & \text{if } 2\notin N_{E/\mathbf{Q}}(\dot{E}) \text{ and } -1\in N_{E/\mathbf{Q}}(\dot{E}),\\ 1, & \text{otherwise.}\end{cases}$$

Of course, when $m<0$, the condition in $(*)$ can be replaced simply by $2\in N_{E/\mathbf{Q}}(\dot{E})$. Note also that under condition $(**)$, $g^+(\mathcal{L})=2^{t-1}$ in all cases.

6. Remark. Under certain conditions on the lattice L and quadratic field E, it was shown in [EH3] that $g^+(L)\leqslant g^+(\mathcal{L})$ for a nonmodular lattice L. The techniques used above can be applied to give an upper bound on $g^+(\mathcal{L})$. For any $\mathbf{Z}$-lattice L with $\dim L\geqslant 3$, there are at most finitely many rational primes p for which $\theta(O^+(L_p))\neq\mathfrak{A}_p\dot{\mathbf{Q}}_p^2$. Similarly, if T denotes the set of nonarchimedean spots on E, the set

$$\mathfrak{S}(\mathcal{L})=\{\mathfrak{p}\in T\colon \theta(O^+(\mathcal{L}_\mathfrak{p}))\neq\mathfrak{U}_\mathfrak{p}\dot{E}_\mathfrak{p}^2\}$$

is also finite. Of interest here is the subset of $\mathfrak{S}(\mathcal{L})$ defined by

$$\mathfrak{S}'(\mathcal{L})=\{\mathfrak{p}\in\mathfrak{S}(\mathcal{L})\colon \theta(O^+(\mathcal{L}_\mathfrak{p}))\neq\dot{E}_\mathfrak{p}\}.$$

Let r denote the number of elements in $\mathfrak{S}'(\mathcal{L})$. Note that here r can be no greater than twice the number of rational primes dividing the determinant of the form. We have

$$\begin{aligned}(J_E\colon P_{\mathfrak{D}}J_E^{\mathcal{L}}) &= (J_E\colon P_EJ_E^{\mathcal{L}})(P_EJ_E^{\mathcal{L}}\colon P_{\mathfrak{D}}J_E^{\mathcal{L}})\\ &\leqslant (J_E\colon P_EJ_E^{e})(P_EJ_E^{e}\colon P_E(J_E^{\mathcal{L}}\cap J_E^{e}))(P_EJ_E^{\mathcal{L}}\colon P_{\mathfrak{D}}J_E^{\mathcal{L}})\\ &\leqslant 2^{t-1}\cdot 2^r\cdot 2^2=2^{r+t+1}.\end{aligned}$$

In the particular case of a unimodular L, r equals 0, thus giving the bound which can be obtained from the table.

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