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A COMPLETE KÄHLER METRIC OF POSITIVE CURVATURE ON $\mathbf{C}^n$

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ABSTRACT. A complete Kähler metric of positive curvature on $\mathbf{C}^n$ is constructed and its importance is discussed.

The purpose of this paper is to exhibit an example of a complete Kähler metric on $\mathbf{C}^n$ with strictly positive Riemannian sectional curvature. To accomplish this, let $r^2 = \sum_{i=1}^n z_i \bar{z}_i$ on $\mathbf{C}^n$ and consider metrics on $\mathbf{C}^n$ of the form $g_{i\bar{j}} = \partial^2 f(r^2)/\partial z_i \partial \bar{z}_j$, $r \to f(r^2) \in C^\infty(R)$. We shall describe the conditions on f which make $g_{i\bar{j}}$ a complete metric of positive curvature, and then show that if $f(x) = \int_0^x (\ln(1+\tau)/\tau)\, d\tau$, these conditions are satisfied, so for this f, $g_{i\bar{j}}$ is a C^∞ complete Kähler metric of strictly positive curvature on $\mathbf{C}^n$.

Previously, there have been no known examples of complete noncompact Kähler manifolds of positive sectional curvature. Nevertheless several theorems have been proved regarding the structure of such manifolds. In particular, such manifolds are Stein manifolds [3], they are real diffeomorphic to R^{2n} [5], and they admit no nonconstant bounded holomorphic functions [6]. For these and other reasons, they have been conjectured to be biholomorphic to $\mathbf{C}^n$ [4]. Thus, the existence of a complete Kähler metric of positive sectional curvature on $\mathbf{C}^n$ is not surprising; but it is not trivial. One can easily show that $\mathbf{C}$ admits a complete Kähler metric of positive curvature: By the existence of isothermal coordinates [2] any complete metric on R^2 of positive sectional curvature is a Hermitian metric of positive sectional curvature relative to some complex structure; the resulting complex manifold is in fact $\mathbf{C}$ as a consequence of the Blanc Fiala Theorem [1] or of the general result quoted on the nonexistence of nonconstant bounded holomorphic functions. If one then takes products of $\mathbf{C}$ with itself one obtains a complete Kähler metric of nonnegative sectional curvature and positive Ricci curvature on $\mathbf{C}^n$. However, there is no obvious way to perturb this metric to obtain a complete Kähler metric of positive sectional curvature, for there is difficulty in finding a perturbation which gives a merely Hermitian metric of positive sectional curvature and, in addition, there is the difficulty of maintaining the Kähler condition, which is given, in effect, by a differential equation. Here, the problem of satisfying the Kähler condition is solved by considering only

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metrics derived from a global potential function, but to check the positivity of the sectional curvatures requires complicated curvature calculations.

Clearly,

$$g = f'(r^2)\left(\sum_{i=1}^{n} dz_i d\bar{z}_i\right) + f''(r^2)\left(\sum_{i=1}^{n} \bar{z}_i dz_i\right)\left(\sum_{i=1}^{n} z_i d\bar{z}_i\right),$$

so g is positive definite if $f'(r^2) > 0$ and $f'(r^2) + r^2 f''(r^2) > 0$ for all r, and g is complete if the integral $\int_0^\infty \sqrt{f'(r^2) + r^2 f''(r^2)}\, dr$ diverges. In addition, g is spherically symmetric so we may restrict the curvature computation for g to the complex line $L = \{z_i = 0 | i > 1\}$ without loss of generality. Since g is a Kähler metric, a messy but elementary computation on L gives

$$\begin{aligned} R_{j\bar{k}l\bar{m}} &= f''(r^2)(\delta_{j\bar{k}}\delta_{l\bar{m}} + \delta_{j\bar{m}}\delta_{l\bar{k}}) \\ &\quad + r^2\left[f'''(r^2) - \frac{(f''(r^2))^2}{f'(r^2)}\right](\delta_{j\bar{k}1}\delta_{l\bar{m}} + \delta_{j\bar{m}1}\delta_{l\bar{k}} + \delta_{l\bar{m}1}\delta_{j\bar{k}} + \delta_{l\bar{k}1}\delta_{j\bar{m}}) \\ &\quad + \left[r^4 f''''(r^2) - r^2\frac{(2f''(r^2) + r^2 f'''(r^2))^2}{f'(r^2) + r^2 f''(r^2)} + 4r^2\frac{(f''(r^2))^2}{f'(r^2)}\right]\delta_{j\bar{k}l\bar{m}1}. \end{aligned}$$

On the set

$$S = \operatorname{span}\{\partial/\partial z_i, \partial/\partial \bar{z}_k | i, k > 1\} \subset T\mathbf{C}^n_L,$$

$$R_{j\bar{k}l\bar{m}} = f''(r^2)(\delta_{j\bar{k}}\delta_{l\bar{m}} + \delta_{j\bar{m}}\delta_{l\bar{k}}) \quad \text{and} \quad g_{i\bar{j}} = f'(r^2)\delta_{i\bar{j}},$$

so g has constant holomorphic sectional curvature $-2f''(r^2)/f'(r^2)$. Therefore, if $f''(r^2) < 0$ for all r, the Riemannian curvatures of all real two planes contained in S will be positive. By spherical symmetry we may assume the remaining two planes to be spanned by two real vectors of the form

$$X = a\frac{\partial}{\partial z_1} + \bar{a}\frac{\partial}{\partial \bar{z}_1}, \quad Y = b\frac{\partial}{\partial z_1} + \bar{b}\frac{\partial}{\partial \bar{z}_1} + c\frac{\partial}{\partial z_2} + \bar{c}\frac{\partial}{\partial \bar{z}_2}, \qquad a, b, c \in \mathbf{C},$$

and by the above

$$K(X, Y) = \frac{(a\bar{b} - \bar{a}b)^2\left[2f''(r^2) + 4r^2 f'''(r^2) + r^4 f''(r^2) - r^2\dfrac{(2f''(r^2) + r^2 f'''(r^2))^2}{f'(r^2) + r^2 f''(r^2)}\right] - 2a\bar{a}c\bar{c}\left[f''(r^2) + r^2 f'''(r^2) - r^2\dfrac{(f''(r^2))^2}{f'(r^2)}\right]}{-(a\bar{b} - \bar{a}b)^2(f'(r^2) + r^2 f''(r^2))^2 + 4a\bar{a}c\bar{c}f'(r^2)(f'(r^2) + r^2 f''(r^2))}.$$

Here $(a\bar{b} - \bar{a}b)^2 < 0$ as $a\bar{b} - \bar{a}b$ is imaginary, and

$$2f''(r^2) + 4r^2f'''(r^2) + r^4f''''(r^2) - r^2\frac{\left(2f''(r^2) + r^2f'''(r^2)\right)^2}{f'(r^2) + r^2f''(r^2)}$$
$$= \frac{f'(r^2) + r^2f''(r^2)}{4r}\ \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln\left(f'(r^2) + r^2f''(r^2)\right)\right).$$

Thus the conditions for $\partial^2 f(r^2)/\partial z_i \partial \bar{z}_j$ to be a complete Kähler metric on $\mathbf{C}^n$ of strictly positive curvature are:

(a) $f'(r^2) + r^2f''(r^2) > 0$,

(b) $\int_0^\infty \sqrt{f'(r^2) + r^2f''(r^2)}\ dr$ diverges,

(c) $f''(r^2) < 0$,

(d)
$$f''(r^2) + r^2f'''(r^2) - r^2\frac{\left(f''(r^2)\right)^2}{f'(r^2)} < 0, \quad \text{and}$$

(e)
$$\frac{1}{4r}\ \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln\left(f'(r^2) + r^2f''(r^2)\right)\right) < 0.$$

Consider now the function

$$f(r^2) = \int_0^{r^2}\frac{\ln(1 + X)}{X}\ dX \in C^4(R).$$

Here

$$f'(X) = \frac{\ln(1 + X)}{X}, \qquad f''(X) = \frac{X/(1 + X) - \ln(1 + X)}{X^2},$$

and

$$f'(r^2) + r^2f''(r^2) = (1 + r^2)^{-1},$$

so f satisfies conditions (a), (b) and (c). Also,

$$f''(r^2) + r^2f'''(r^2) - r^2\frac{\left(f''(r^2)\right)^2}{f'(r^2)} = \frac{\ln(1 + r^2) - r^2}{r^2(1 + r^2)^2\ln(1 + r^2)} < 0$$

and

$$\frac{1}{4r}\ \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln\left(f'(r^2) + r^2f''(r^2)\right)\right) = \frac{-1}{(1 + r^2)^2} < 0,$$

so conditions (d) and (e) are satisfied.

Thus for $f(r^2) = \int_0^{r^2}(\ln(1 + x)/x)\,dx$, $g = \partial^2 f(r^2)/\partial z_i \partial \bar{z}_j$ is a complete Kähler metric of strictly positive sectional curvature on $\mathbf{C}^n$.

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