

TWO NOTES ON METRIC GEOMETRY

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ABSTRACT. If μ , $\int d\mu = 1$, is a signed Borel measure on the unit ball in E^3 , it is shown that $\sup_{\mu} \int \int |p - q| d\mu(p) d\mu(q) = 2$ with no extremal measure existing. Also, a class of simplices which generalizes the notion of acute triangle is studied. The results are applied to prove inequalities for determinants of the Cayley-Menger type.

1. **A counterexample.** In this article μ will always denote a signed Borel measure, concentrated on a compact set K in Euclidean space, which satisfies $\int d\mu = 1$. Let $I(\mu)$ denote the integral $\int \int |p - q| d\mu(p) d\mu(q)$. It was shown in [1] and [2] that if $K \subset E^n$, then there is a constant b_n such that $\sup_{\mu} I(\mu) < b_n$ (diameter of K). It was conjectured that an extremal measure μ_0 would always exist so that $I(\mu_0) = \sup_{\mu} I(\mu)$. This conjecture was known to be true in case K was a finite set or the surface of a ball.

However, we now show that an integral formula developed in [2], together with Archimedes' beautiful theorem on zonal areas, allows us to construct a counterexample when K is a solid ball in E^3 .

Let us assume that the ball K has radius 1, and let N be a large positive number. Place a uniform measure of total mass N on the surface of K , and place a uniform negative measure of total mass $1 - N$ on the surface of the concentric ball of radius $1 - N^{-1}$. The measure μ_N is defined to be the sum of these two uniform measures.

A key observation is that if t_1 and t_2 are two parallel planes which both cut the ball of radius $1 - N^{-1}$, then μ_N assigns measure zero to the portion of K between t_1 and t_2 . This follows from Archimedes' theorem.

Let τ be the usual invariant measure on the planes of E^3 such that the measure of the set of planes which cut a line segment is the length of the segment. We shall use without proof the fact that the measure of the set of planes in E^3 which cut a ball of radius R is $4R$.

If t is a plane which cuts K , let A_t and B_t be the portions of K lying in the respective open half-spaces determined by t . In [2] it was established that

$$(1) \quad I(\mu) = 2 \int \mu(A_t) \mu(B_t) d\tau(t).$$

Also, it was shown that for almost all t , $\mu(A_t) + \mu(B_t) = 1$, and hence for almost all t , $\mu(A_t) \mu(B_t) \leq \frac{1}{4}$.

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LEMMA 1. *There is no signed Borel measure of total mass 1, concentrated on the unit ball K , for which $\mu(A_t) = \mu(B_t) = \frac{1}{2}$ for almost all t .*

PROOF. Suppose such a measure exists. Let a plane t_0 cut K so that the volume of A_{t_0} is $\varepsilon > 0$. The set of planes t for which $A_t \subset A_{t_0}$ has positive τ -measure, and hence $\mu(A_t) = \frac{1}{2}$ for at least one such t . Thus $|\mu|(A_{t_0}) \geq \frac{1}{2}$. Since this is true for any $\varepsilon > 0$, we quickly deduce by measure continuity from above that each point on the surface of K is an atom of $|\mu|$ -measure at least $\frac{1}{2}$. This cannot be true since $|\mu|(K) < \infty$ by the definition of signed measure. (See the book of Halmos [5, Chapter 6].)

The lemma together with (1) allows us to conclude that if μ_0 exists then $I(\mu_0) < 2 \cdot 4 \cdot \frac{1}{4} = 2$. The measure of the planes t which cut the ball of radius $1 - N^{-1}$ is $4(1 - N^{-1})$ and for any such t , $\mu_N(A_t)\mu_N(B_t) = \frac{1}{4}$ by Archimedes' theorem. If t is a plane which does not cut the ball of radius $1 - N^{-1}$, then certainly $\mu(A_t)\mu(B_t) \geq 0$. Thus

$$I(\mu_N) \geq 2[4(1 - N^{-1})]/4 = 2(1 - N^{-1}).$$

By choosing N large enough, $I(\mu_N) > I(\mu_0)$, a contradiction. We have proved the following theorem.

THEOREM 1. *Let K be the unit ball in E^3 . Then $\sup_{\mu} I(\mu) = 2$, but no extremal measure μ_0 exists.*

We remark that the same type of theorem can be proved when K is allowed to range over the family of sets of constant width 2 in E^3 . Thus if K is any compact set of diameter not exceeding 2 and μ is concentrated on K , $I(\mu) < 2$, and this is the best possible. The corresponding question for positive measure is open. The paper of Björck [3] contains much information about $I(\mu)$, restricted to the class of positive measures.

2. Acute simplices and Cayley-Menger determinants. If K is a finite set in Euclidean space, the functional $I(\mu)$ achieves an absolute maximum for a unique, necessarily atomic, measure μ_0 as was shown in [1]. Also, it was shown that

$$(2) \quad I(\mu_0) = 2R^2$$

where R is the radius of the least sphere which circumscribes the metric space $(K, d^{1/2})$ viewed as a subset of Euclidean space. The work of I. J. Schoenberg [6] assures us that if d is the Euclidean metric, then $(K, d^{1/2})$ embeds isometrically into E^m as the vertices of a nondegenerate m -simplex if $|K| = m + 1$. The number R is the radius of the $(m - 1)$ -sphere which contains the points of K .

DEFINITION 1. We call a simplex in E^m with distinct vertices p_0, p_1, \dots, p_m *r*-acute if there exists a simplex in E^r with vertices q_0, q_1, \dots, q_m such that $|p_i - p_j|^2 = |q_i - q_j|^2$ for all i, j .

The number r is not unique, but generally the least possible value would be chosen. The previously mentioned work of Schoenberg assures us that the

acute simplex of the definition must be a nondegenerate m -simplex. Our next theorem says much more. Before stating the theorem we point out the obvious fact that the altitude from a vertex p_i of a simplex does not exceed $\min_j |p_i - p_j|$, $i \neq j$.

THEOREM 2. *Let S be an r -acute simplex. Then there is an absolute positive constant c_r such that the altitude from any vertex p_i is not less than*

$$c_r \min_j |p_i - p_j|, \quad i \neq j.$$

LEMMA 2. *Let p'_1, \dots, p'_m be the images of p_1, \dots, p_m under the inversion through a sphere of radius 1 centered at p_0 . Then p'_0, p'_1, \dots, p'_m ($p'_0 = p_0$) are the vertices of an r -acute simplex S' if and only if p_0, \dots, p_m are the vertices of an r -acute simplex S .*

PROOF. Let q_0, \dots, q_m be points in E^r for which $|q_i - q_j| = |p_i - p_j|^2$. First suppose that neither i nor j is 0, and that q'_0, q'_1, \dots, q'_m are the respective images of q_0, \dots, q_m under the inversion through the unit sphere centered at $q_0 = q'_0$. We note that

$$|q'_i - q'_j| = |q_i - q_j| [|q_i - q_0| |q_j - q_0|]^{-1} = |p'_i - p'_j|^2.$$

Also $|q'_0 - q'_i| = |q_0 - q_i|^{-1} = |p'_0 - p'_i|^2$.

The proof in the opposite direction is symmetrical.

PROOF OF THEOREM 2. Using the notation of Lemma 2, we see that the altitude h from p_0 in the simplex S is precisely D^{-1} where D is the diameter of the circumsphere (in E^m) of the simplex S' . This follows at once from the fact that the inversion at p_0 sends the hyperplane spanned by p_1, \dots, p_m into a sphere which contains p_0 , and that the point in the hyperplane nearest to p_0 will be sent to a point diametrically opposite to p_0 on the circumsphere of S' .

Next we use known results about the functional $I(\mu)$ to obtain an upper bound on D , which will give a lower bound on $h = D^{-1}$. The measure μ will be defined by $\mu(q'_i) = x_i$ with $\sum x_i = 1$ so that

$$I(\mu) = \sum |q'_i - q'_j| x_i x_j = \sum |p'_i - p'_j|^2 x_i x_j.$$

We know that there is an absolute constant b_r such that

$$(3) \quad I(\mu) \leq b_r \{ \max_{i,j} |q'_i - q'_j| \} \quad \text{for any } \mu.$$

This result is found in [1, Theorem 3.8]. Combining (3) and (2) we obtain

$$(4) \quad \frac{1}{2} D^2 \leq b_r \{ \max_{i,j} |p'_i - p'_j|^2 \}.$$

Since $\max_{i,j} |q'_i - q'_j| \leq 2 \max_i |q'_0 - q'_i|$, we have $D^2 \leq 4b_r \max_i |p'_0 - p'_i|^2$. Since $|p'_0 - p'_i| = |p_0 - p_i|^{-1}$, it follows that $D^{-1} = h \geq c_r \min_i |p_0 - p_i|$ where $c_r = (2b_r^{1/2})^{-1}$. This completes the proof. It may be true that $c_r = \sqrt{2}/2$ will work in all dimensions.

As an application of the theorem we prove an inequality for Cayley-Menger type determinants.

COROLLARY 1. Let q_0, q_1, \dots, q_n be points in a Euclidean space, then there is an absolute positive constant d_n such that

$$(5) \quad \left| \det \begin{pmatrix} 0 & 1 \\ 1 & |q_i - q_j| \end{pmatrix} \right| > d_n [\min_{i \neq j} |q_i - q_j|]^n.$$

PROOF. We consider an n -acute simplex S whose vertices p_0, \dots, p_n satisfy $|p_i - p_j|^2 = |q_i - q_j|$. Any subsimplex of an acute simplex is certainly an acute simplex. Also, any altitude from any vertex of a subsimplex of S will satisfy $h > c_n \min_{i \neq j} |p_i - p_j|$ by Theorem 2. Thus $\text{Vol}(S)$ is at least $(n!)^{-1} [c_n \min_{i \neq j} |p_i - p_j|]^n$.

It is well known (see [4, Chapter 4]) that

$$\left| \det \begin{pmatrix} 0 & 1 \\ 1 & |p_i - p_j|^2 \end{pmatrix} \right| = d'_n [\text{Vol}(S)]^2.$$

Substituting the previously mentioned lower bound for $\text{Vol}(S)$ completes the proof with $d_n = d'_n (n!)^{-2} c_n^{2n}$.

Although the Cayley-Menger type determinant arises most frequently, $\det(|q_i - q_j|)$ is also of interest. We note that this determinant equals

$$\left(\prod_{i=1}^n |q_0 - q_i|^2 \right) \det \begin{pmatrix} 0 & 1 \\ 1 & |q'_i - q'_j| \end{pmatrix}$$

where $|q'_i - q'_j|$ are the inverted distances previously discussed. Thus, if need be, we can also bound this determinant away from zero.

In closing, we note that inequality (3) can also be easily derived from formula (1), letting $K = \{q'_0, \dots, q'_m\}$. We would have, using the fact that any set of diameter D is contained in a ball of radius D ,

$$(6) \quad I(\mu) < 2\left(\frac{1}{4}\right)(\text{diameter } K)\tau_r,$$

where τ_r is the measure of the hyperplanes in E^r which cut a unit ball in E^r .

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