ORIENTATION-REVERSING PERIODIC PL MAPS
OF LENS SPACES
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Abstract. We complete the classification of all orientation-reversing PL
maps of period $4k$ ($k > 1$) on lens spaces. All $\mathbb{Z}_4$-actions on the projective
3-space are also classified.

1. Introduction. The 3-sphere and the projective 3-space are the only
3-dimensional lens spaces $L(p, q)$ which admit orientation-reversing PL maps
of period $n > 4$ [4]. In [4] all orientation-reversing PL maps of period $4k$ on
the 3-sphere $S^3$ are classified for all $k$. In this paper we show:

Theorem A. The projective 3-space $\mathbb{P}^3$ admits a unique orientation-reversing
PL map of period $4k$ for each $k$, up to conjugation.

All orientation-reversing PL involutions of lens spaces [6], [8] and all PL
involutions of $\mathbb{P}^3$ [2], [6] are known. As consequences of Theorem A and [3],
we have

Theorem B. The projective 3-space $\mathbb{P}^3$ admits exactly four distinct $\mathbb{Z}_4$-actions
(PL), up to conjugation.

The cyclic group generated by $h$ shall be denoted by $\langle h \rangle$. Two actions of
$\langle h \rangle$ and $\langle h' \rangle$ on a space $M$ are said to be conjugate if there exists a
homeomorphism $t$ of $M$ such that $\langle tht^{-1} \rangle = \langle h' \rangle$. We shall denote the
fixed-point set of $h$ by $\text{Fix}(h)$.

We consider $S^3$ as a subset of $\mathbb{C}^2$, defined by $\{(z_1, z_2) \in \mathbb{C}^2 | z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\}$. Define an orientation-reversing homeomorphism $\lambda'$ of $S^3$ by $\lambda'(z_1, z_2) = (\omega z_1, \omega z_2)$, where $\omega = e^{2\pi i/n}$ and $n$ is even. Since $\lambda'$ commutes with the
antipodal map of $S^3$, there exists an obvious orientation-reversing map $\lambda$ of
period $n$ on $\mathbb{P}^3$ induced by $\lambda'$.

Remark. Let $f$ be an orientation-preserving map of period $n$ on $S^3$ and
$\text{Fix}(f) \neq \emptyset$. Then $\text{Fix}(f)$ is a simple closed curve. A well-known conjecture,
due to P. A. Smith, asserts that $\text{Fix}(f)$ is unknotted for all $n$ (see [1]). It
follows from a result of F. Waldhausen [12] that the conjecture is true for
even period $n$. Let $h$ be an orientation-reversing map of period $n > 2$ on $\mathbb{P}^3$.
Then it follows from the Lefschetz fixed-point theorem that $\text{Fix}(h) \neq \emptyset$. Note
that $h^2$ is orientation-preserving and $n$ is even. Assuming the Smith conjecture, one can prove that $\pi_1(P^3 - \text{Fix}(h^2))$ is abelian. The following theorem classifies all orientation-reversing periodic maps of $P^3$ modulo the Smith conjecture.

**Theorem C.** Let $h$ be an orientation-reversing PL map of the projective 3-space $P^3$ with period $n \geq 2$. If $\pi_1(P^3 - \text{Fix}(h^2))$ is abelian, then $h$ is conjugate to $\lambda$.

Let $h$ be a periodic map of a space $M$. Then there exists a homeomorphism $q$ of $M/\langle h^k \rangle$, uniquely determined by $h$, such that $qg = gh$, where $g: M \to M/\langle h^k \rangle$ is the orbit map. We call $q$ the map of $M/\langle h^k \rangle$ induced by $h$. We shall denote the closed unit interval and the $n$-sphere by $I$ and $S^n$, respectively. In this paper all spaces and maps will be in the PL category.

2. **Equivariant product structure.** Let $A$ be the annulus $S^1 \times I$ and let $h$ be a free involution on $A \times I$ such that $h(A \times \{i\}) = A \times \{i\}$ ($i = 0, 1$). The following lemma is essentially a special case of a result of Waldhausen [11].

**Lemma 2.1.** There exists an isotopy of $A \times I$, keeping $A \times \{i\}$ invariant, after which a product structure on $A \times I$ and an involution $g$ on $A$ are obtained so that $h(x, t) = (g(x), t)$ for $(x, t) \in A \times I$. The isotopy can be chosen constant on $A \times \{0\}$.

Suppose that $A \times I$ is a regular neighborhood of an invariant simple closed curve $J$, or equivalently $\pi_1(A \times I - J) = \mathbb{Z} \oplus \mathbb{Z}$. The following lemma further claims that the isotopy in Lemma 2.1 can be chosen, after which $J = S^1 \times \{1/2\} \times \{1/2\}$.

**Lemma 2.2.** There exists an isotopy of $A \times I$, keeping $A \times \{i\}$ invariant, after which a product structure on $A \times I$ and an involution $g$ on $A$ are obtained so that $h(x, t) = (g(x), t)$ for $(x, t) \in A \times I$ and $J = S^1 \times \{1/2\} \times \{1/2\}$.

**Proof.** We will show that there exists an invariant annulus $H$ properly embedded in $A \times I$ such that $\partial H$ meets each $S^1 \times \{i\} \times (0, 1)$ ($i = 1, 0$) in a simple closed curve and $J \subset \text{Int} H$. Then $H$ separates $A \times I$ into two invariant components, each of which is homeomorphic to $S^1 \times I \times I$. Define a product structure on each component $S^1 \times I \times I$ as in Lemma 2.1. We may assume that $J = S^1 \times \{1/2\} \subset S^1 \times I \approx H$. Now repairing the cut along $H$ defines the required product structure of $A \times I$.

Let $M = A \times I/\langle h \rangle$ and $g: A \times I \to M$ be the orbit map. Then $M$ is a homeomorphic to $c \times I$, where $c$ is either an annulus or a möbius band. It is not difficult to see that $M$ is a regular neighborhood of $g(J)$. Therefore, there exists a properly embedded surface $H'$ in $M$ such that each component of $\partial M - (g(A \times \{0, 1\}))$ meets $H'$ in a simple closed curve, where $H'$ is a regular neighborhood of $g(J)$ in $H'$ (see Lemma 2.1). Then $H = g^{-1}(H')$ is an annulus as desired.
3. Proof of Theorem A.

(3.1) Consider an orientation-reversing map $h$ of period $4k$ on $P^3$. It follows from the Lefschetz fixed-point theorem that $\text{Fix}(h) \neq \emptyset$. Since $h^{2k}$ is an involution of $P^3$, $\text{Fix}(h^{2k})$ is a disjoint union of two simple closed curves, say $F$ and $F'$ (see [2]). Let $N = P^3/\langle h^{2k} \rangle$ and $d: P^3 \to N$ be the orbit map. Let $L = d(F)$ and $L' = d(F')$. Then $N$ is homeomorphic to $S^3$ and

$$\pi_1(N - L - L') = \mathbb{Z} \oplus \mathbb{Z}$$

(see [2]). Let $f$ be the map of $N$ induced by $h$. Since $f$ is an orientation-reversing map of $N$ (of period $2k$) and $d(\text{Fix}(h)) \subset \text{Fix}(f)$, $\text{Fix}(h)$ consists of two points, say $x$ and $\bar{x}$. Let $y = d(x)$ and $\bar{y} = d(\bar{x})$. We may assume that \{x, $\bar{x}$\} $\subset$ $F'$. Then $\text{Fix}(f) = \{y, \bar{y}\} \subset L'$. Notice that if $k > 1$, $\text{Fix}(f^{2r})$ ($1 < r < k$) is a simple closed curve. Hence $\text{Fix}(h^{2r})$ is a simple closed curve, and it is easy to see that $\text{Fix}(h^{2r}) = F'$ for each $r, r \not\equiv 0$ (mod $k$). Let $M = N/\langle h^{2r} \rangle$, $g: N \to M$ be the orbit map (of course, $M = N$ if $k = 1$). Then $M$ is again homeomorphic to $S^3$ (see [3]). Let $T$ be the map of $M$ induced by $f$. Then $T$ is an orientation-reversing involution. Let $J = g(L), J' = g(L'), \bar{z} = g(\bar{y}),$ and $z = g(y)$. Notice that $\text{Fix}(T) = \{z, \bar{z}\} \subset J'$ and $T$ interchanges the two open arcs $J' - \{z, \bar{z}\}$. Since $\pi_1(N - L - L') = \mathbb{Z} \oplus \mathbb{Z}$, it can be seen that $\pi_1(M - J - J') = \mathbb{Z} \oplus \mathbb{Z}$ (see [3]).

(3.2) A product structure will be defined on the complement of a regular neighborhood of $J'$ in $M$. Take small invariant balls $B$ and $\bar{B}$ in $M - J$, containing $z$ and $\bar{z}$, respectively, such that $B \cap \bar{B} = \emptyset$. Again take a small invariant regular neighborhood of $\partial(J - B - \bar{B})$ in $\partial(M - B - \bar{B}) - J$ so that it has two components, say $K$ and $K'$. Let $\tilde{M} = B \cup \bar{B} \cup K \cup K'$. Then $\tilde{M}$ is a regular neighborhood of $J'$. Let $\overline{M} = \partial(M - \tilde{M})$. Since

$$\pi_1(M - J - J') = \mathbb{Z} \oplus \mathbb{Z},$$

it follows from a result of J. Stallings [10] that $\overline{M}$ is homeomorphic to a solid torus $D^2 \times S^1$ and it is also an invariant regular neighborhood of $J$. Parametrize $\overline{M}$ in terms of $A \times I$ ($A = S^1 \times I$) such that

$$\partial A \times I \approx \partial(\partial(K \cup K') - B - \bar{B}), \quad A \times \{0\} \approx \partial B - k - K'$$

and

$$A \times \{1\} \approx \partial B - K - K'.$$

(3.3) Let $h_i$ ($i = 1, 2$) be an orientation-reversing map of period $4k$ on $P^3$. In connection with $h_i$, a symbol such as $q_i$ shall be used to represent the same object as $q$ where $q$ is a symbol in (3.1) and (3.2). We will define an equivalence $\iota$ of $M_1$ onto $M_2$ such that $T_2 \iota = \iota T_1$, $\iota(J_i) = J_2$, and $\iota(J'_i) = J'_2$. Since $J$ is isotopic to $S^1 \times \{1/2\} \times \{1/2\}$ in $A \times I$ (after parametrizing $\overline{M}$), it follows from Lemma 2.2 that there exists an equivalence $\iota$ between $T_1|\overline{M}_1$ and $T_2|\overline{M}_2$ such that $\iota(J_i) = J_2$, $\iota(\partial B_i - K_i - K'_i) \equiv \partial B_2 - K_2 - K'_2$, and $\iota(\partial B_i - K_i - K'_i) \equiv \partial B_2 - K_2 - K'_2$. Since each $T_i$ interchanges $K_i$ and $K'_i$, and $T_i|B_i (T_i|B_i \text{ resp.})$ is essentially the cone over $T_i|\partial B_i (T_i|\partial B_i \text{ resp.})$ (see
one can extend \( \tilde{t} \) to an equivalence \( t \) between \( T_1 \) and \( T_2 \) such that
\[
\tilde{t}(J_1) = J_2.
\]

(3.4) Since \( \pi_1(N_i - L_i) = Z \) and \( \text{Fix}(f_i^{2r}) = L_i, 1 < r < k \), there exists a lifting equivalence \( s \) between \( f_1 \) and \( f_2 \) such that \( g_2^{-1} = g_2 \) (see the following diagram). Let \( \tilde{N}_i = g_i^{-1}(M_i) \) and \( \tilde{P}_i = d_i^{-1}(\tilde{N}_i) \). Then \( \tilde{N}_i \) and \( \tilde{P}_i \) are invariant regular neighborhoods of \( L_i \) and \( \tilde{F}_i \) which are homeomorphic to \( D^2 \times S^1 \), respectively. Our plan is to find a lifting \( r \) of \( P^3 \) such that the following diagram commutes.

\[
\begin{array}{ccc}
(P^3, \tilde{P}_1, F_1) & \xrightarrow{d_1} & (N_1, \tilde{N}_1, L_1) \\
\downarrow r & & \downarrow s \\
(P^3, \tilde{P}_2, F_2) & \xrightarrow{d_2} & (N_2, \tilde{N}_2, L_2) \\
\end{array}
\]

To do this, it is enough to show that \( (s'D') \# (\pi_1(\tilde{P}_i)) = d_i^{-1} \# (\pi_1(\tilde{P}_i)) \), where \( s' : \partial \tilde{N}_1 \to \partial \tilde{N}_2 \) and \( d' : \partial \tilde{P}_i \to \partial \tilde{N}_i \) are the maps such that \( s' = s|_{\partial \tilde{N}_i} \) and \( d' = d_i|_{\partial \tilde{P}_i} \). In (3.5) and (3.6) we select some special generators of \( \pi_1(\partial \tilde{P}_i) \). In (3.7) we conclude the proof.

(3.5) Let \( q_i \) be the orbit map of \( T_i \) and let \( Q_i = q_i(M_i) \). Since \( t \) is an equivalence between \( T_1 \) and \( T_2 \), there exists a homeomorphism \( \hat{t} \) between the orbit spaces of \( T_1 \) and \( T_2 \) such that \( \hat{t} q_i = q_2 t \). Let \( q_i(J_i) = J_2 \). It follows from Lemma 2.2 that \( Q_i \) is a nonorientable disk bundle over \( J_i \) (see also (3.2)). Therefore one can find an annulus \( A_i \) in \( Q_i \) such that \( \text{Int}(A_i) \subset \text{Int}(Q_i) \) and \( \partial A_i \) consists of \( J_i \) and a simple closed curve \( c_i \) on \( \partial Q_i \). Let \( A_2 = t(A_i) \). Then \( A_2 \) is an annulus in \( Q_2 \) such that \( \text{Int}(A_2) \subset \text{Int}(Q_2) \) and \( \partial A_2 \) consists of \( J_2 \) and a simple closed curve \( c_2 \) on \( \partial Q_2 \).

(3.6) Let \( v_i = q_i \cdot (g_i|\tilde{N}_i) \). Then, since \( v_i : \tilde{N}_i \to Q_i \) is the covering projection generated by \( f_i|\tilde{N}_i \), we see that \( v_i^{-1} \) is an annulus in \( \tilde{N}_i \) such that \( \partial (v_i^{-1}(A_i)) \) consists of \( L_i \) and a simple closed curve \( v_i^{-1}(c_i) \) on \( \partial \tilde{N}_i \), and \( \text{Int}(v_i^{-1}(A_i)) \subset \text{Int}(\tilde{N}_i) \) (recall that \( L_i \) is invariant under \( f_i \)). On the other hand, there exists a properly embedded disk \( D_i \) in \( \tilde{P}_i \) such that \( D_i \cap h_i^{2r}(D_i) = \emptyset \) \((1 < r < k)\), \( h_i^{2k}(D_i) = D_i \), and \( \partial D_i \) does not bound a disk in \( \partial \tilde{P}_i \) (letting \( h_i = h_i|\tilde{P}_i \), then \( h_i^{2k} \) is the only nontrivial element of \( \langle h_i \rangle \) with nonempty fixed point set, and one may use the argument in the proof of Lemma 2.8 of [3]). Let \( \alpha_i \) be the element of \( \pi_1(\partial \tilde{P}_i) \) represented by the path \( \partial D_i \) and \( \beta_i \) be another element of \( \pi_1(\partial \tilde{P}_i) \) such that \( \alpha_i \) and \( \beta_i \) generate \( \pi_1(\partial \tilde{P}_i) \). Let \( \gamma_i \) and \( \xi_i \) be the elements of \( \pi_1(\partial \tilde{N}_i) \) represented by \( d_i(\partial D_i) \) and \( v_i^{-1}(c_i) \). Since \( d_i(\partial D_i) \) bounds a disk in \( \tilde{N}_i \) and \( v_i^{-1}(c_i) \) is homotopic to the center circle of \( N_i \) \((\approx D^2 \times S^1)\), we see that \( \gamma_i \) and \( \xi_i \) generate \( \pi_1(\partial \tilde{N}_i) \). Since \( v_i^{-1}(c_i) \) is connected, where \( v_i = v_i|\partial D_i \). Suppose that it were disconnected. Recall that \( d_i' = d_i|\partial \tilde{P}_i \). Since \( d_i' \) is a double covering projection generated by \( h_i^{2k}|\partial \tilde{P}_i \) and \( v_i^{-1}(c_i) \) is connected, we see that \( v_i^{-1}(c_i) \)
has two components, say \( a_i, b_i \), and, hence, \( h^{2k}(a_i) = b_i \). Since \( \tilde{\omega} \partial \tilde{P}_i \) is a covering projection generated by \( h_i \partial \tilde{P}_i \), \( a_i \cup b_i \) is invariant under \( h_i \) (and \( h^k \)), and either \( h_i(a_i) = b_i \) or \( h_i(a_i) = a_i \) occurs. Therefore we see that \( h^{2k}(a_i) = a_i \), which is a contradiction to the fact that \( h^{2k}(a_i) = b_i \). Hence \( \tilde{\omega}^{-1}(c_i) \) is connected. This implies that there is no element \( \phi_i \) of \( \pi_1(\partial \tilde{P}_i) \) such that 
\[ d^i_\#(\phi_i) = \xi_i. \]
We may assume (by the choice of \( D_i \) in \( P_i \)) that 
\[ d^i_\#(\tau_i) = \gamma_2 \] 
and 
\[ d^i_\#(\beta_i) = \gamma^m \xi_i \] 
for some \( m_i \). Note that \( m_i \) must be odd (otherwise, \( \xi_i \in d^i_\#(\pi_1(\partial \tilde{P}_i)) \)). Therefore, \( d^i_\#(\pi_i(\partial \tilde{P}_i)) \) is generated by \( \gamma_2^2 \) and \( \gamma_2 \xi_i \). Since 
\[ s^i_\#(\gamma_i^2) = \gamma_2^2 \] 
and 
\[ s^i_\#(\gamma_2 \xi_i) = \gamma_2 \xi_2 \]
we have a lifting \( \tilde{r} \) of \( P^3 - F_1 - F_2 \) to \( P^3 - F_1 - F_2 \). One may extend \( \tilde{r} \) to a homeomorphism \( r \) of \( P^3 \) such that \( d_2 r = s_1 d_1 \). This completes the proof.

4. Proof of Theorem C.

(4.1) Consider an orientation-reversing map \( h \) of period \( n \) on \( P^3 \). In §3 we have proved the case where \( n = 4k \). Now assume that \( n = 2k, k \) odd \( > 1 \).

Since \( h^k \) is an involution on \( P^3 \), \( \text{Fix}(h^k) \) is a projective plane \( P \) plus an isolated point \( x \) (see [6]). Since the period of \( h^2 \) is \( k \) (odd), \( \text{Fix}(h^2) \) (1 < \( r < k \)) is a simple closed curve (see [3]). Let \( F = \text{Fix}(h^2) \). Let \( \xi : S^3 \to P^3 \) be the natural projection. Since \( \pi_1(P^3) = Z_2 \), it can be seen that \( \xi^{-1}(F) \) is a simple closed curve [3], [7]. Since \( \pi_1(P^3 - F) \) is abelian, \( \pi_1(S^3 - \xi^{-1}(F)) = Z \) (see [9]). Therefore, \( \pi_1(P^3 - F) = Z \) (for a proof, see [7]).

(4.2) Since \( \pi_1(P^3 - F) = Z \) and \( k \) is odd, the orbit space \( M = P^3/\langle h^2 \rangle \) is homeomorphic to \( P^3 \) (see [3]). Since \( P \cup \{ x \} \) is invariant under \( h \), we see that \( h(x) = x \). Since \( \text{Fix}(h) \subset F \), \( \text{Fix}(h) \) consists of \( x \) and a point \( y \) of \( F \). Since \( h^2 \) and \( h^k \) generate the group \( \langle h \rangle \), \( F \cap P = \{ y \} \). Furthermore, since \( h \) interchanges the sides of \( P \) in a small neighborhood of \( y \), \( F \) meets \( P \) at \( y \) locally piercingly. Let \( g : P^3 \to M \) be the orbit map and let \( J = g(F) \). Since \( \pi_1(P^3 - F) = Z \), it can be seen that \( \pi_1(M - J) = Z \) (see [3]). Let \( T \) be the involution of \( M \) induced by \( h \). Since \( gh^k = Tg \), we see that \( \text{Fix}(i) = g(P) \cup g(x) \). Let \( \tilde{P} = g(P) \) and \( z = g(x) \). Then \( \tilde{P} \) is a projective plane and \( T \) interchanges the two open arcs of \( J - \{ z, g(y) \} \). Note that \( \tilde{P} \) is one-sided in \( M \).

(4.3) Triangulate \( M \) so that \( \text{Fix}(T) \) and \( J \) are subcomplexes, and \( T \) becomes simplicial. Let \( U \) be the simplicial neighborhood of \( \tilde{P} \) in \( M \). Let \( B \) be the closed star of \( z \) in \( M \). We may assume that \( B \cap U = \emptyset \). Note that \( T|B \) is essentially a cone over \( T|\partial B \) (see [8]). Consider the double covering \( \gamma : M' \to M \) obtained from \( M \) by cutting along \( \tilde{P} \). Since \( \partial U \approx S^2 \), we see that \( cl(M - U) \) is homeomorphic to a 3-cell, and \( M' \) is a 3-sphere \( S^3 \). Since \( \pi_1(M - J) = Z \), we see that \( \pi_1(M' - J') = Z \) where \( J' = \gamma^{-1}(J) \). Therefore \( J' \) is unknotted in \( M' \) (see [9]).

(4.4) Now consider the two orientation-reversing maps \( h_1 \) and \( h_2 \) of \( P^3 \) with period \( 2k, k \) odd \( > 1 \). As in (3.3), we use symbols \( q_i \) in connection with \( h_i \) (\( i = 1, 2 \)) whenever a symbol \( q \) has appeared in (4.2). Let 
\[ Q_i = cl(M_i - U_i - B_i). \]
Let $K_i$ be the simplicial neighborhood of $\text{cl}(J_i - U_i - B_i)$ in $Q_i$ such that $K_i$ has two components. Since $\gamma_i^{-1}(U_i)$ is a product neighborhood $S_i \times [-1, 1]$ of $\gamma_i^{-1}(P_i) \approx S^2$ such that $S_i \times (0) = \gamma_i^{-1}(P_i)$, each component of $\text{cl}(M'_i - \gamma_i^{-1}(U_i \cup B_i))$ is homeomorphic to $S^2 \times I$ and it is exactly a copy of $Q_i$. Therefore, since $J_i$ is unknotted in $M_i$, a product structure on $\text{cl}(Q_i - K_i)$ can be defined in terms of $A \times I (A = S^1 \times I)$ such that $\text{cl}(\partial B_i - K_i) \approx A \times \{0\}$, $\text{cl}(\partial U_i - K_i) \approx A \times \{1\}$, and $\text{cl}(\partial K_i - B_i - U_i) \approx S^1 \times \{0, 1\} \times I$. Furthermore the product structure can be chosen so that there exists an involution $f$ on $A$ such that $T_i(x, t) = (f(x), t)$ for $(x, t) \in A \times I$ (see §2). Notice that $T_i$ interchanges the two components of $K_i$. Therefore, the orbit space of $T_i|\text{cl}(M_i - B_i - U_i)$ is homeomorphic to $P^2 \times I$ (see also [8]) and, letting $G_i = M_i/\langle T_i \rangle$ and $k_i: M_i \to G_i$ be the orbit map, there exists a homeomorphism $\alpha$ of $G_1$ to $G_2$ such that $(ak_1)(z_1) = k_2(z_2)$, $(ak_1)(J_1) = k_2(J_2)$, and $(ak_1)(P_1) = k_2(P_2)$. Hence there exists an equivalence $\beta$ between $T_1$ and $T_2$ such that $\beta(J_1) = J_2$. Since $\pi_1(M_i - J_i) = Z$, one may conclude by the lifting theorem that $h_1$ and $h_2$ are conjugate in the usual way.

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