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A NOTE ON QUILLEN'S PAPER "PROJECTIVE MODULES OVER POLYNOMIAL RINGS"[1]

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ABSTRACT. We give a simplified proof to the following theorem due to D. Quillen: if A is a commutative noetherian ring of global dimension $\leqslant 1$, then finitely generated projective modules over $A[T_1, \ldots, T_n]$ are extended from A. We prove also that if A is a commutative noetherian ring of global dimension d, then finitely generated projective modules of rank $> d$ over $A[T_1, \ldots, T_n]$ are extended from A.

Let A be a commutative ring. As in [1] an $A[T_1, \ldots, T_n]$-module M will be called *extended* from A if it is isomorphic to $A[T_1, \ldots, T_n] \otimes_A N$ for some A-module N; in this case $N \cong M/(T_1, \ldots, T_n)M$. We use here Theorem 1′ of [1]:

QUILLEN'S LOCALIZATION THEOREM. *Let M be a finitely presented module over $A[T_1, \ldots, T_n]$. If $M_{\mathfrak{m}}$ is extended from $A_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of A, then M is extended from A.*

THEOREM 1 [1]. *Let A be a commutative noetherian ring of global dimension $\leqslant 1$. Then finitely generated projective modules over $A[T_1, \ldots, T_n]$ are extended from A.*

PROOF. By Theorem 1′ of [1] we can assume A is local and so we have to prove that finitely generated projective modules over $R = A[T_1, \ldots, T_n]$ are free. We prove this assertion by induction on n starting with $n = 0$. Let $n \geqslant 1$. By [2, Corollary (22.3)] it is enough to prove that R is Hermite: that is if P is a finitely generated R-module and $P \oplus R$ is free then P is free. Let P be such an R-module. Let $\{l_1, \ldots, l_m\}$ be a free basis of $P \oplus R$, let π be the canonical projection π: $P \oplus R \to R$ and $\pi(l_i) = a_i$ $(1 \leqslant i \leqslant m)$.

$Ra_1 + \cdots + Ra_m = R$ and we have to show that the row $[a_1, \ldots, a_m]$ is completable to an invertible matrix over R. We can assume $m \geqslant 3$. As Krull dimension of A is $\leqslant 1$, we can assume by [3, §4, Lemme 3] that a_m is a unitary polynomial in T_n. Let $S = A[T_1, \ldots, T_{n-1}]$. By [1, Theorem 1] and the induction hypothesis it is enough to show that $P_{\mathfrak{m}}$ is a free $S_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of S. $P_{\mathfrak{m}} = \ker \pi_{\mathfrak{m}}$ and so for a given $\mathfrak{m}$ we have to show that the row $[\bar{a}_1, \ldots, \bar{a}_m]$ is completable to an invertible matrix over $S_{\mathfrak{m}}[T_n]$.

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($\bar{a}_i$ is the canonical image of a_i in $S_{\mathfrak{m}}$.) Now $S_{\mathfrak{m}}[T_n]/\bar{a}_n S_{\mathfrak{m}}[T_n]$ is a semilocal ring as $S_{\mathfrak{m}}$ is local and $\bar{a}_m$ is unitary in T_n. Therefore there are only a finite number of maximal ideals $M_1, \ldots, M_r$ in $S_{\mathfrak{m}}[T_n]$ which contain $\bar{a}_m$. There are $\lambda_2, \ldots, \lambda_{m-1}$ in $S_{\mathfrak{m}}[T_n]$ such that $a_1' = \bar{a}_1 + \lambda_2\bar{a}_2 + \cdots + \lambda_{m-1}\bar{a}_{m-1}$ does not belong to M_i for any i and so a_1' and $\bar{a}_m$ generate the ideal (1) of $S_{\mathfrak{m}}[T_n]$. Therefore the given row is completable to an invertible matrix over $S_{\mathfrak{m}}[T_n]$ and $P_{\mathfrak{m}}$ is free.

REMARKS. We proved above that any unimodular row over $A[T]$ (A any commutative ring) which contains a unitary polynomial is completable to an invertible matrix over $A[T]$ provided any unimodular row over A is completable to an invertible matrix over A (that is A is Hermite). In fact the last assumption is superfluous by Theorem 3 of [1].

The theorem above includes Theorems 4 and 4′ of [1]. It can be obtained also as a direct consequence of Theorems 1′ and 4 of [1].

The argument above gives us

THEOREM 2 (CF. [2, THEOREM (22.1)]). *Let A be a commmutative noetherian ring of finite global dimension d. Then finitely generated projective modules of rank $> d$ over $A[T_1, \ldots, T_n]$ are extended from A.*

Both theorems above are particular cases of Bass' problem quoted in [1, (1)].

REMARK. Professor L.N. Vaseršteĭn (Moscow) has informed the author that Serre's problem was independently solved by A. Suslin (Leningrad). The simple argument above seems to be known to him. He has also communicated to the author a simple solution of Serre's problem which does not use Quillen's localization theorem.

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