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THE k-CLOSURE OF MONIC AND MONIC FREE IDEALS IN A POLYNOMIAL SEMIRING

LOUIS DALE

ABSTRACT. The concepts of k-closure, k-boundary and weak k-ideals are introduced and necessary and sufficient conditions that an ideal be a k-ideal are given. These conditions are applied to monic and monic free k-ideals. Also, it is shown that the ascending chain condition holds for monic ideals, but not for monic free ideals, and that a semiring S is Noetherian if and only if $S[x]$ satisfies the ascending chain condition for monic ideals.

1. **Introduction.** Monic and monic free ideals in a polynomial semiring were introduced in [1] and structure theorems were given for monic ideals and monic free k-ideals in a strict polynomial semiring. In this paper, the theory of monic and monic free ideals will be applied to the concept of the k-closure of an ideal and necessary and sufficient conditions that an ideal be a k-ideal will be given. The concepts of k-boundary and weak k-ideal are introduced. Also, monic and monic free ideal theory will be applied to Noetherian semirings and the Hilbert basis theorem.

2. **Fundamentals.** The concepts of half-ring, nearring, hemiring and semiring all appear in the literature, and what is universally true about them is that they are all something "less" than a ring. For this paper, a semiring will be a set S together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(S, +)$ is an abelian semigroup with a zero, $(S, \cdot)$ is a semigroup, and multiplication distributes over addition from both the left and the right. A semiring S is said to be commutative if $(S, \cdot)$ is a commutative semigroup and S is said to have an identity if there exists $1 \in S$ such that $1 \cdot x = x \cdot 1 = x$ for each $x \in S$. A semiring S is said to be a strict semiring if $a, b \in S$ and $a + b = 0$ imply $a = b = 0$. Ideals in semirings are defined in the usual manner.

Let S be a commutative semiring with an identity and $S[x]$ be the semiring of polynomials over S in the indeterminate x.

2.1. DEFINITION. An ideal M in $S[x]$ will be called monic if $\sum a_i x^i \in M$ implies $a_i x^i \in M$ for each $i \in \{0, 1, \ldots, n\}$.

2.2. DEFINITION. An ideal F in $S[x]$ will be called monic free if M is a monic ideal such that $M \subset F$, then $M = \{0\}$.

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2.3. Definition. An ideal I in a semiring S will be called a k-ideal if $a \in I, b \in S$ and $a + b \in I$ imply $b \in I$.

Examples of monic ideals, monic free ideals and k-ideals are found in [1]. Throughout this paper, unless otherwise stated, S will be a commutative semiring with an identity and $S[x]$ will be the semiring of polynomials over S in the indeterminate x.

3. **Monic and monic free k-closures.** In a ring, every ideal is a k-ideal. However, not every ideal in a semiring is a k-ideal but every ideal is contained in a k-ideal. Thus it is reasonable to give consideration to the k-closure of an ideal.

3.1. Definition. Let A be an ideal in a semiring S. The ideal

$$\overline{A}_k = \bigcap \{B \mid B \text{ is a } k\text{-ideal and } A \subset B\}$$

will be called the k-closure of A.

Since the semiring S is itself a k-ideal, it follows that $\overline{A}_k \neq \varnothing$ for any ideal A in S. By definition, it is always the case that $A \subset \overline{A}_k$. If A is a k-ideal, then $\overline{A}_k \subset A$ and it follows that $A = \overline{A}_k$. Consequently, an ideal A is a k-ideal if and only if $A = \overline{A}_k$. It is clear that $\overline{A}_k$ is the smallest k-ideal containing A.

Now let A be an ideal in $S[x]$. In order for A to be a k-ideal, A must have the following property: given polynomials $f \in A$ and $g \in S[x]$ such that $f + g \in A$, it follows that $g \in A$. It may happen that A is not a k-ideal but has this property for all polynomials of a certain degree. Thus, if A is not a k-ideal one can try to determine the largest integer n, if it exists, such that A is a k-ideal with respect to all $f \in A$ with degree $f \leqslant n$. If no such n exists, one can look for the smallest ideal B containing all $f \in A$ with degree $f \leqslant n$, such that B is a k-ideal with respect to polynomials of degree less than or equal to n. This is formalized in the following definitions.

3.2. Definition. An ideal A in $S[x]$ will be called a weak k-ideal if there exists an integer n such that A is a k-ideal with respect to all $f \in A$ with degree $f \leqslant n$. The largest such integer, if it exists, will be called the k-degree of A. If no such integer exists, then A is said to have k-degree ∞.

It is clear that if A is a k-ideal, then A must have k-degree ∞. Also, every k-ideal is a weak k-ideal, but not every weak k-ideal is a k-ideal.

3.3. Definition. Let A be an ideal in $S[x]$, n a fixed integer, and

$$A_{f_n} = \{f \mid f \in A \text{ and degree } f \leqslant n\}.$$

The ideal

$$\overline{A}_{k_n} = \bigcap \{B \mid B \text{ is a weak } k\text{-ideal with } k\text{-degree at least } n \text{ and } A_{f_n} \subset B\}$$

will be called the weak k-closure of A, or the k_n-closure of A.

Since the k-closure of A has k-degree greater than n, it follows that $\overline{A}_{k_n} \neq \varnothing$. It is clear that $\overline{A}_{k_n}$ is a weak k-ideal with k-degree greater than or equal to n. For any ideal A, $\{A_{f_n}\}$ is an ascending chain of sets and $A = \bigcup A_{f_n}$.

Consequently, $\{\bar{A}_{k_n}\}$ is an ascending chain of ideals and $\lim \bar{A}_{k_n} = \bigcup \bar{A}_{k_n}$. While it is true that an ideal A is always contained in its k-closure, it is not generally true that A is contained in its k_n-closure. Since $A_{f_n} \subset \bar{A}_{k_n}$, it is clear that (A_{f_n}), the ideal generated by A_{f_n}, is contained in $\bar{A}_{k_n}$.

The proof of the next lemma requires the use of the following lemma whose proof may be found in [1].

3.4. LEMMA. *If A is a k-ideal in $S[x]$, $f = a_n x^n + \cdots + a_0 \in A$, and τ is a nonnegative integer, then*

$$(a_n x^n + \cdots + a_{i+1}x^{i+1} + a_{i-1}x^{i-1} + \cdots + a_0)^{2\tau+1} + (a_i x^i)^{2\tau+1} \in A.$$

3.5. LEMMA. *Let S be a strict semiring and A be an ideal in $S[x]$. If A is a monic free k-ideal, then $(A_{f_n}) \neq A$, (A_{f_n}) is a weak k-ideal with finite k-degree, and $(A_{f_n}) = \bar{A}_{k_n}$.*

PROOF. Recall that (A_{f_n}) is generated by A_{f_n}. Let $\Delta\colon S[x] \to Z^+$ be a function defined as follows: (1) if $h = b_n x^n + b_{n-p}x^{n-p} + \cdots + b_0$ has degree n and $b_{n-p} \neq 0$, then $\Delta(h) = p$; (2) if $h = bx^n$, $b \neq 0$, then $\Delta(h) = n$; and (3) if $h = 0$, then $\Delta(h) = 0$. Since A is monic free, each $g \in A_{f_n}$ contains at least two nonzero terms and, consequently, $1 \leqslant \Delta(g) \leqslant n$. Now suppose that

$$f = a_n x^n + a_{n-p}x^{n-p} + \cdots + a_0 \in A_{f_n}.$$

Clearly $\Delta(f) = p$ and it follows from Lemma 3.4 that $f_\tau = (a_n x^n)^{2\tau+1} + (a_{n-p}x^{n-p} + \cdots + a_0)^{2\tau+1} \in A$ for each $\tau \in Z^+$. It is clear that $\Delta(f_\tau) = (2\tau + 1)p$. Since $p \geqslant 1$ is fixed and $\tau = 0, 1, 2, \ldots,$ the sequence $\{(2\tau + 1)p\}$ is an increasing sequence of integers. Consequently, there is a λ such that $\Delta(f_\lambda) = (2\lambda + 1)p > n$. If $f_\lambda \in (A_{f_n})$, then

$$(1)\quad f_\lambda = (a_n x^n)^{2\lambda+1} + (a_{n-p}x^{n-p} + \cdots + a_0)^{2\lambda+1} = h_1 g_1 + \cdots + h_m g_m$$

where $h_i \in S[x]$ and $g_i \in A_{f_n}$ with $\Delta(g_i) = c_i \leqslant n$. At least one of these products, say $h_i g_i$, must produce a term of degree $(2\lambda + 1)n$, since $(a_n x^n)^{2\lambda+1}$ appears on the left side of (1). From $\Delta(g_i) = c_i$, it follows that

$$g_i = b_m x^m + b_{m-c_i}x^{m-c_i} + \cdots + b_0.$$

Moreover, h_i must contain a term of degree $(2\lambda + 1)n - m$. Now multiplication of g_i by this term produces a term of degree $(2\lambda + 1)n$ and a term of degree $(2\lambda + 1)n - c_i$. Since S is a strict semiring, none of these terms can vanish. Consequently, the right side of (1) contains a term of degree $(2\lambda + 1)n - c_i$. Since $(2\lambda + 1)p > n \geqslant c_i$, it follows that

$$(2)\quad \begin{aligned}(2\lambda + 1)n &> (2\lambda + 1)n - c_i \geqslant (2\lambda + 1)n - n \\ &> (2\lambda + 1)n - (2\lambda + 1)p = (2\lambda + 1)(n - p).\end{aligned}$$

The second highest degree term on the left side of (1) is $(2\lambda+1)(n-p)$. Hence a term of degree $(2\lambda+1)n-c_i$ cannot appear on the left side of (1) because of (2), a contradiction. Therefore $f_\lambda \notin (A_{f_n})$ and $(A_{f_n}) \neq A$. Let $h, h+g \in (A_{f_n})$ and suppose that $\deg h \leqslant n$ and $\deg(h+g) \leqslant n$. Since A is a k-ideal, it follows that $g \in A$. Now S being a strict semiring implies that $S[x]$ is a strict semiring. Consequently, $\deg g \leqslant \deg(h+g) \leqslant n$, since no terms of the sum $h+g$ can sum up to zero. Therefore $g \in A_{f_n}$ and (A_{f_n}) is a weak k-ideal. Next, let $h = a_n x^n$, $g = a_{n-p}x^{n-p} + \cdots + a_0$ and $f_\tau = h^{2\tau+1} + g^{2\tau+1}$ with $f_0 \in A_{f_n}$. It was shown above that there exists a λ such that $f_\lambda \notin (A_{f_n})$. Now

$$(f_0)^3 = h^3 + 3h^2g + 3hg^2 + g^3 = h^3 + g^3 + 3hg(h+g) = f_1 + (3hg)f_0.$$

Since $f_0, (f_0)^3 \in (A_{f_n})$ and A is a k-ideal, $f_1 \in A$. Now either $f_1 \in (A_{f_n})$ or $f_1 \notin (A_{f_n})$. If $f_1 \notin (A_{f_n})$, then (A_{f_n}) has finite k-degree. If $f_1 \in (A_{f_n})$, then consider

$$(f_0)^5 = h^5 + g^5 + (5hg)f_1 + (10h^2g^2)f_0 = f_2 + (5hg)f_1 + (10h^2g^2)f_0.$$

Since $f_0, f_1, (f_0)^5 \in (A_{f_n})$ and A is a k-ideal, $f_2 \in A$. Now either $f_2 \in (A_{f_n})$ or $f_2 \notin (A_{f_n})$. If $f_2 \notin (A_{f_n})$, then (A_{f_n}) has finite k-degree. If $f_2 \in (A_{f_n})$, then consider $(f_0)^7$ and continue as above. This process must stop since there exists a λ such that $f_\lambda \notin (A_{f_n})$. If $\deg f_\lambda = t$, then the k-degree of (A_{f_n}) is less than t. Since A is a k-ideal, it is clear that (A_{f_n}) is a weak k-ideal with k-degree at least n. Now (A_{f_n}) is the smallest weak k-ideal containing A_{f_n} and it follows that $(A_{f_n}) = \overline{A}_{k_n}$.

It is worthwhile to note that the k_n-closure of an ideal is not unique. That is, it is possible that $\overline{A}_{k_n} = \overline{A}_{k_m}$ when $m \neq n$.

3.6. Lemma. *If A is an ideal in $S[x]$, then* $\lim \overline{A}_{k_n} = \overline{A}_k$.

Proof. Let $\lim \overline{A}_{k_n} = K$. Since $\{\overline{A}_{k_n}\}$ is an ascending chain of ideals and $K = \bigcup \overline{A}_{k_n}$, it follows that K is an ideal. Now if f and $f+g \in K$ with degrees m and n, respectively, then f and $f+g \in \overline{A}_{k_t}$, where $t = \max\{m, n\}$. Consequently, $g \in \overline{A}_{k_t} \subset K$ and K is a k-ideal. From $A_{f_n} \subset \overline{A}_{k_n} \subset K$ for each n, it follows that $A = \bigcup A_{f_n} \subset \bigcup \overline{A}_{k_n} = K$. Now K being a k-ideal containing A and $\overline{A}_k$ being the smallest k-ideal containing A, together imply that $\overline{A}_k \subset K$. Also, $\overline{A}_{k_n} \subset \overline{A}_k$ for each n, and it follows that $K = \bigcup \overline{A}_{k_n} \subset \overline{A}_k$. Consequently, $\overline{A}_k = K$ and the results follow.

At this point, recall from [1] that if F is a monic free k-ideal in $S[x]$, S a strict semiring, then every basis for F is infinite. Also, if F is a monic k-ideal, then F necessarily has a basis B consisting of elements of the form ax^i, $a \in S$. If S is a Noetherian semiring (that is, S satisfies the ascending chain condition on ideals), then B is finite even though $S[x]$ is not necessarily Noetherian. These remarks help in the proof of the following theorem.

3.7. THEOREM. *Let S be a semiring. An ideal A in $S[x]$ is a k-ideal if and only if* $\lim \overline{A}_{k_n} = A$. *If A is a monic free k-ideal and S is strict, then $\{\overline{A}_{k_n}\}$ is an infinite, proper ascending chain. If A is a monic k-ideal and S is Noetherian, then $\{\overline{A}_{k_n}\}$ is a finite chain.*

PROOF. If A is a k-ideal, then $A = \overline{A}_k = \lim \overline{A}_{k_n}$. On the other hand, if $\lim \overline{A}_{k_n} = A$, then it follows from Lemma 3.6 that A is a k-ideal. Now if A is a monic free k-ideal and S is strict, then Lemma 3.5 assures that for any n, $(A_{f_n}) \neq A$ and $(A_{f_n}) = \overline{A}_{k_n}$. Hence, there exists a $g \in A$ such that $\deg g = p > n$ and $g \notin (A_{f_n})$. Consequently, (A_{f_n}) is properly contained in (A_{f_p}) and it follows that $\{\overline{A}_{k_n}\}$ is an infinite, proper ascending chain. Next, suppose that A is a monic k-ideal and S is Noetherian. Then A has a finite basis B consisting of polynomials of the form ax^i. Let n be the maximum of the degrees of these polynomials and consider A_{f_n}. Since A is a k-ideal and $B \subset A_{f_n}$, it follows that $A = (A_{f_n}) = \overline{A}_{k_n}$, and $\{\overline{A}_{k_n}\}$ is finite.

If A is not a k-ideal, a method of determining the k-closure of A is needed. If it is possible to determine the collection of all k-ideals containing A, then it only remains to find the intersection of this collection. If this is not possible, then consideration should be given to the k-boundary of A.

3.8. DEFINITION. Let E be an ideal in a semiring S. The set

$$E' = \{x \in S \mid \text{there is an } a \in E \text{ such that } a + x \in E\}$$

is called the k-boundary of E.

3.9. THEOREM. *If E is an ideal in S, then E' is a k-ideal in S and $E \subset E'$.*

PROOF. Given x_1 and $x_2 \in E'$ there are elements a_1 and $a_2 \in E$ such that $a_1 + x_1 \in E$ and $a_2 + x_2 \in E$. Since $a_1 + a_2 \in E$ and

$$(a_1 + x_1) + (a_2 + x_2) = (a_1 + a_2) + (x_1 + x_2) \in E,$$

it follows that $(x_1 + x_2) \in E'$. If $b \in S$, then $b(a_1 + x_1) = ba_1 + bx_1 \in E$. Consequently, $bx_1 \in E'$ and E' is an ideal. Now suppose that $x \in E', y \in S$ and $x + y \in E'$. Then there exists u and $v \in E$ such that $x + u \in E$ and $(x + y) + v \in E$. Now $(x + u) + v \in E$ and $[(x + y) + v] + u \in E$. But $[(x + y) + v] + u = [(x + u) + v] + y$. Consequently, $y \in E'$ and E' is a k-ideal. If $e \in E$, then $0 + e \in E$ and it follows that $e \in E'$ and $E \subset E'$.

3.10. COROLLARY. *An ideal E in S is a k-ideal if and only if $E = E'$.*

PROOF. If E is a k-ideal, it is clear that $E' \subset E$ and Theorem 3.9 assures that $E = E'$. On the other hand, if $E = E'$, Theorem 3.9 assures that E is a k-ideal.

3.11. COROLLARY. *If E is an ideal in S, then $E' = \overline{E}_k$.*

PROOF. Theroem 3.9 gives that E' is a k-ideal containing E. Consequently, $\overline{E}_k \subset E'$. But any k-ideal containing E has to contain the k-boundary of E. Thus $E' \subset \overline{E}_k$ and it follows that $E' = \overline{E}_k$.

Now given an ideal A in $S[x]$, the k-closure of A can be found by computing the k-boundary of A or by finding the intersection of all of the k-ideals containing A. The first method may be thought of as an internal method while the second may be thought of as an external method. If A is a weak k-ideal with finite k-degree, it may be of interest to determine the k-degree of A. This may be done by an inspection of a basis for A and in some cases, making use of Lemma 3.5.

3.12. EXAMPLES. (i) Let $a, b \in Z^+$, $a > 1$, $b > 1$, $a < b$, and $b \notin (a)$. Consider the ideal $A = (a, b, x + b)$ in $Z^+[x]$ generated by a, b, and $x + b$. This ideal is not a k-ideal since $x + b \in A$ and $b \in A$, but $x \notin A$. By the division algorithm, $b = pa + c$ for some $c < a$. Hence $pa + c \in A$ and $pa \in A$, but $c \notin A$. Consequently, there is no nonnegative integer n such that A is a k-ideal with respect to all polynomials in A of degree less than or equal to n. Therefore A is not a weak k-ideal. The k-boundary of A can be found by including x and the greatest common divisor of a and b, say d, in the basis for A. Thus $A' = (a, b, d, x, x + b) = (d, x)$. In this case, it is clear that A' is a k-ideal in $Z^+[x]$ and $A' = \bar{A}_k$.

(ii) Let $b \in Z^+$ and $b > 1$. Consider the ideal $B = (b, x^n + b)$, where $n > 1$. This ideal is not a k-ideal in $Z^+[x]$, since $x^n \notin B$. But B is a weak k-ideal of degree $n - 1$. To see this, note that any polynomial of degree m, where $m < n$, is of the form $\sum a_i(bx^i)$ and B is certainly a k-ideal with respect to these polynomials, since each $a_i(bx^i) \in B$. The k-boundary of B is found by including x^n in the basis for B. Consequently, $B' = (b, x^n, x^n + b) = (b, x^n) = \bar{B}_k$.

(iii) Let $A_0 = (x^2 + 1)$, $A_n = (x^{4n+2} + 1) + A_{n-1}$ if $n > 0$ and $A = \bigcup A_n$. It was shown in [1] that A is a monic free k-ideal in $Z^+[x]$. Now $A_{f_1} = \varnothing$ and $A_{f_2} = \{b(x^2 + 1)\}$ since A contains no polynomials of degree 1 and $x^2 + 1$ is the only polynomial of degree 2 in the basis for A. Since A is a monic free k-ideal, Lemma 3.5 assures that $(A_{f_2}) = \bar{A}_{k_2} = (x^2 + 1)$ is a weak k-ideal of finite k-degree. Clearly $x^6 + 1 \neq g(x)(x^2 + 1)$ for any $g(x) \in Z^+[x]$ and it follows that $x^6 + 1 \notin \bar{A}_{k_2}$. But $(x^2 + 1)^3 = (x^6 + 1) + 3x^2(x^2 + 1) \in \bar{A}_{k_2}$ and $3x^2(x^2 + 1) \in \bar{A}_{k_2}$. Consequently, the k-degree of $\bar{A}_{k_2}$ is less than six. It is clear that any polynomial in A of degree t, where $t \leqslant 5$, is in the ideal $\bar{A}_{k_2} = (x^2 + 1)$. Therefore the k-degree of $\bar{A}_{k_2}$ is 5. Also, it follows that $\bar{A}_{k_2} = \bar{A}_{k_3} = \bar{A}_{k_4} = \bar{A}_{k_5}$. Again, since A is a k-ideal $\bar{A}_{k_6} = (x^2 + 1) + (x^6 + 1)$ is a weak k-ideal of finite k-degree and $\bar{A}_{k_2} \subset \bar{A}_{k_6}$. It can be shown in the same manner as above that the k-degree of $\bar{A}_{k_6}$ is 9. In general, the k-degree of $\bar{A}_{k_{4n+2}}$ is $4n + 5$. Continuing in this manner, one can obtain a proper infinite ascending chain of weak k-ideals $\{\bar{A}_{k_n}\}$, each with finite k-degree, whose limit is A.

These examples illustrate that an ideal in a polynomial semiring may be a k-ideal, a weak k-ideal, or neither.

4. **Monic and monic free ideals and Hilbert's theorem.** The Hilbert basis theorem provides a clear contrast between monic and monic free ideals. If S

is a Noetherian semiring, will $S[x]$ satisfy the ascending chain condition for monic ideals? Will $S[x]$ satisfy the ascending chain condition for monic free ideals? The answer to the first question is yes while the answer to the second is no. To see this, note that any monic free k-ideal in a polynomial semiring $S[x]$ over a strict semiring S can be written as a proper infinite ascending chain of ideals. This remains true even though the semiring S may be Noetherian. Consequently, $S[x]$ cannot have the ascending chain condition for monic free ideals. A consequence of this is that the Hilbert basis theorem is not true in general for semirings. Thus, the class of monic free k-ideals in a polynomial semiring over a Noetherian strict semiring will provide the necessary counter-examples. For example, Z^+ is a Noetherian strict semiring. It is obvious that Z^+ is a strict semiring. A proof that Z^+ is Noetherian may be found in [2]. But $Z^+[x]$ is not a Noetherian semiring since the ideals $\{A_n\}$, where $A_0 = (x^2 + 1)$ and $A_n = (x^{4n+2} + 1) + A_{n-1}$, if $n > 1$, is a proper infinite ascending chain of ideals in $Z^+[x]$. Now in the case of monic ideals, not only will $S[x]$ satisfy the ascending chain condition for monic ideals, but the converse is also true. Before proving this, a few preliminaries are necessary.

Let A be an ideal in $S[x]$ and

$$A_i = \{a \in S \mid \text{there is } f \in A \text{ such that } ax^i \text{ is a term of } f\}.$$

It is easy to show that $\{A_i\}$ is an ascending chain of ideals in S. These ideals are called coefficient ideals.

4.1. Lemma. *If A and B are monic ideals in $S[x]$, then $A = B$ if and only if $A_i = B_i$ for each i.*

Proof. If $A = B$ and $f(x) \in A$, then any coefficient a_i of f is in both A_i and B_i and, consequently, $A_i = B_i$. Conversely, suppose $A_i = B_i$ for each i and $f = \sum a_i x^i \in A$. Then $a_i \in B_i$ for each i and there are polynomials $f_i \in B$ such that $a_i x^i$ is a term of f_i. Since B is a monic ideal, it follows that each $a_i x^i \in B$. Consequently, $f = \sum a_i x^i \in B$ and $A \subset B$. Similarly, $B \subset A$ and it follows that $A = B$.

4.2. Theorem. *A semiring S is Noetherian if and only if every ascending chain of monic ideals in $S[x]$ is finite.*

Proof. Let S be Noetherian and $\{A_n\}$ be an ascending chain of monic ideals in $S[x]$. Consider the corresponding coefficient ideals $\{A_{nt}\}$ in S. It is clear that these ideals form a double array of ascending ideals, i.e. if $i < n$ and $j < m$, then $A_{ij} \subseteq A_{nm}$. Now consider the diagonal $\{A_{kk}\}$ of this array. These ideals form an ascending chain and S being Noetherian assures that there exists p such that $A_{kk} = A_{pp}$ for all $k \geqslant p$. Let A_{ij} be an ideal such that $i, j \geqslant p$ and let $q = \max\{i,j\}$. Then $p \leqslant i \leqslant q$ and $p \leqslant j \leqslant q$ and it follows that $A_{pp} \subseteq A_{ij} \subseteq A_{qq}$. Consequently, $A_{ij} = A_{pp}$ and it follows that $A_{ij} = A_{kj}$ for all $i, k, j \geqslant p$. Now for each $j < p$, $\{A_{nj}\}$ is an ascending chain and it follows that there exists n_j such that $A_{nj} = A_{n_j j}$ for all $n > n_j$. Now let

$N = \max\{p, n_1, n_2, \ldots, n_{p-1}\}$. Then it is clear that $A_{Nt} = A_{nt}$ for each $n > N$ and $t = 0, 1, 2, \ldots$. Consequently, Lemma 4.1 assures that $A_N = A_m$ for each $m > N$. Conversely, suppose that every ascending chain of monic ideals in $S[x]$ is finite and $\{E_n\}$ is an ascending chain of ideals in S. Now $\{E_n[x]\}$, where

$$E_n[x] = \{f = \sum a_i x^i \in S[x] | a_i \in E_n\},$$

is an ascending chain of monic ideals in $S[x]$. To see this, let $f = \sum a_i x^i$, $g = \sum b_i x^i \in E_n[x]$ and $h = \sum c_i x^i \in S[x]$. Since E_n is an ideal, $(a_i + b_i) \in E_n$ and it follows that $f + g \in E_n[x]$. Now $hf = \sum p_t x^t$, where $p_t = \sum c_i a_j$ for $i + j = t$. Thus $p_t \in E_n$ for each t and, consequently, $hf \in E_n[x]$. It is clear from the definition that $E_n[x]$ is monic. Since $E_n \subset E_{n+1}$, it follows that $E_n[x] \subset E_{n+1}[x]$ and $\{E_n[x]\}$ is an ascending chain of ideals. Consequently, there exists N such that $E_N[x] = E_m[x]$ for each $m > N$. Now consider the coefficient ideals $(E_n[x])_i$ of $E_n[x]$. From the definition of $E_n[x]$, it follows that $(E_n[x])_i = E_n$ for each i. Lemma 4.1 assures that $(E_N[x])_i = (E_m[x])_i$ for each i. Therefore $E_N = (E_N[x])_i = (E_m[x])_i = E_m$ and S is Noetherian.

The study of monic and monic free ideals in polynomial semirings has given some insight into the structure of ideals in polynomial semirings. Any ideal in a polynomial semiring is either monic, monic free, or mixed. Consequently, the structure of these ideals is important. This was seen while establishing necessary and sufficient conditions for an ideal to be a k-ideal, and while investigating the chain condition for monic and monic free ideals.

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Department of Mathematics, University of Alabama, Birmingham, Alabama 35294