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THE DOMAIN COVERED BY A TYPICALLY-REAL FUNCTION

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ABSTRACT. We find the largest possible domain that is covered by $f(E)$ for every typically-real function $f(z)$. In the process we obtain a set of universal typically-real functions.

1. **A set of universal typically-real functions.** We recall that a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

regular in E: $|z| < 1$, is said to be typically-real in E if $f(z)$ is real for real z and only for real z in E [5]. We let TR denote the set of all functions with the normalization (1) that are typically-real in E.

We first describe a particularly interesting universal[1] typically-real function.

THEOREM 1. *The function*

$$G(z) = \pi^{-1}\tan\big(\pi z/(1 + z^2)\big) \tag{2}$$

is in TR. *In E, the function $G(z)$ assumes each real value exactly once, it omits $\pm i/\pi$, and it assumes every other complex value infinitely often. Further $G'(z)$ is never zero in E.*

PROOF. To visualize the behavior of $G(z)$ we describe the surface $G(E)$. It consists of two infinite sets of half-planes, $\operatorname{Im} w > 0$ and $\operatorname{Im} w < 0$, each set joined by a branch point similar in nature to $\ln(w \pm i/\pi)$. One half-plane of the first set is joined to one half-plane of the second set along the real axis. For each of the other half-planes, the real axis is the boundary.

For simplicity, set $F(z) = \pi G(z)$ and let $\zeta = z/(1 + z^2)$. Then

$$F(z) = \tan \pi\zeta = \frac{\sin \pi\zeta}{\cos \pi\zeta} = \frac{1}{i}\,\frac{e^{\pi i\zeta} - e^{-\pi i\zeta}}{e^{\pi i\zeta} + e^{-\pi i\zeta}} = i\,\frac{1 - e^{2\pi i\zeta}}{1 + e^{2\pi i\zeta}}. \tag{3}$$

Consider first the transformation $\zeta = z/(1 + z^2)$ on the upper half of the unit disk; E^+. It is easy to see that $\zeta(E^+)$ is the half-plane $\operatorname{Im} w > 0$. The interval $[-1, 1]$ goes into the interval $[-1/2, 1/2]$, and the upper boundary of E^+ goes into the remainder of the real axis in a one-to-one manner.

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[1] $H(z)$ is a universal typically-real function (with respect to $w = a + bi$) if every typically-real function that omits $a + bi$ is subordinate to $H(z)$.

Next $\eta = e^{2\pi i\zeta}$ takes this half-plane onto the interior of infinitely many copies of the unit disk, except for the point $\eta = 0$ where the infinitely many copies have a branch point of type $\ln \eta$. In this transformation the interval $-1/2 \leqslant \zeta \leqslant 1/2$ goes into the boundary of the unit disk with $\eta(-1/2) = \eta(1/2) = -1$. Finally $F(z) = i(1 - \eta)/(1 + \eta)$ carries this infinite family of disks onto an infinite family of half-planes, $\operatorname{Im} w > 0$, tied together at $w = i$. The composition of these mappings carries the interval $-1 < z < 1$ onto the real axis on one of the infinitely many half-planes. The Schwarz-reflection principle shows that $F(z)$ carries the full disk E onto a surface of the type described at the beginning of the proof. It is then obvious geometrically that $G(z) \equiv F(z)/\pi$ has all the properties described in Theorem 1. It is easy to give an analytic proof that in E, $G(z) \neq \pm i/\pi$, $G'(z) \neq 0$, and $G(z)$ assumes every other value $a + bi$ $(b \neq 0)$ infinitely often.

The function $G(z)$ can be regarded as a universal function because the surface $G(E)$ will carry $f(E)$ for any other typically-real function that omits $\pm i/\pi$. If $b \neq 0$, then an easy transformation of $F(z)$ (see the next section) gives a universal function for typically-real functions that omit $z = a \pm bi$.

2. **The region covered by** $f(z)$. Brannan and Kirwan [1], proved that for every $f(z)$ in TR, the domain $f(E)$ covers the disk $|z| < 1/4$. We will determine the maximum domain D covered by $f(E)$. It turns out that D is not a disk, but of course it contains the disk $|z| < 1/4$.

We begin with the simple case in which $f(z)$ omits bi in E, where $b > 0$. Throughout our discussion we use the fact that $f(E)$ is symmetric with respect to the real axis for any typically-real function. Hence we can restrict our attention to the upper half-plane $\operatorname{Im} w > 0$.

Let $f(z) \in \mathrm{TR}$ and omit bi, where $b > 0$. Then $f(z)/\pi b$ omits i/π. If $G^{-1}(w)$ denotes the inverse function of $G(z)$, defined on the surface $G(E)$, then

$$B(z) \equiv G^{-1}(f(z)/\pi b) \tag{4}$$

is regular in E and satisfies the conditions of Schwarz's Lemma. Hence $f(z) = \pi b G(B(z))$,

$$f'(z) = \pi b G'(B(z)) B'(z) \tag{5}$$

and at $z = 0$, we have

$$1 = f'(0) = \pi b G'(0) B'(0) \leqslant \pi b, \tag{6}$$

since $G'(0) = 1$ and $0 < B'(0) \leqslant 1$. Further, equality can occur if and only if $B(z) \equiv z$ and $f(z) \equiv G(z)$. Thus if $f(z) \in \mathrm{TR}$ and omits bi with $b > 0$ then $b \geqslant 1/\pi$ and the inequality is sharp.

Now assume that $f(z) \in \mathrm{TR}$ and omits $a + bi$ where $a \neq 0$ and $b > 0$. For convenience set $a + bi = \rho e^{i\alpha}$ where $0 < \alpha < \pi$ and $\alpha \neq \pi/2$ since $a \neq 0$. We let c be the unique real root in E of the equation

$$G(z) = -\pi^{-1} \cot \alpha = -\pi^{-1} a b^{-1}, \tag{7}$$

where $G(z)$ is our universal function (2). Theorem 1 assures us that there is exactly one root in $(-1, 1)$. A brief computation using $\tan(\pi z/(1+z^2)) = \tan(\alpha - \pi/2)$ gives

$$z^2 + (2\pi/(\pi - 2\alpha))z + 1 = 0, \tag{8}$$

and hence

$$c = (-\pi + 2\sqrt{\pi\alpha - \alpha^2})/(\pi - 2\alpha). \tag{9}$$

With this c set

$$H(z) = \pi b G((z+c)/(1+cz)) + a. \tag{10}$$

Then the function $H(z)$ is a universal typically-real function that omits $a + bi$, and $H(0) = 0$. If $H^{-1}(w)$ is the inverse function of $H(z)$ defined on the surface $H(E)$, then $H^{-1}(f(z)) \equiv B(z)$, where $B(z)$ satisfies the conditions of Schwarz's Lemma. Consequently $f(z) = H(B(z))$ and

$$\begin{aligned} 1 = f'(0) &= H'(0)B'(0) = \pi b G'(c)(1 - c^2)B'(0) \\ &= \pi b\left(1 + \tan^2 \frac{\pi c}{1+c^2}\right)\left(\frac{1-c^2}{1+c^2}\right)^2 B'(0), \end{aligned}$$

$$1 = \pi b \csc^2 \alpha \left(\frac{1-c^2}{1+c^2}\right)^2 B'(0). \tag{11}$$

Since c is a root of equation (8) we have

$$\begin{aligned} \left(\frac{1-c^2}{1+c^2}\right)^2 &= \frac{(1+c^2)^2 - 4c^2}{(1+c^2)^2} \\ &= \frac{[2\pi/(\pi - 2\alpha)]^2 - 4}{[2\pi/(\pi-2\alpha)]^2} = \frac{4}{\pi^2}(\pi\alpha - \alpha^2). \end{aligned} \tag{12}$$

We use this in (11), together with $0 < B'(0) \leqslant 1$, and $b \csc\alpha = \rho$. This gives

$$\rho \geqslant \frac{\pi}{4}\,\frac{\sin\alpha}{\alpha(\pi - \alpha)}, \qquad 0 < \alpha < \pi, \alpha \neq \pi/2, \tag{13}$$

with equality if and only if $B(z) \equiv z$. But equality is possible. When $B(z) \equiv z$, then $f(z) = H(z)$.

We note that in the proof we were forced to assume that $a \neq 0$, $\alpha \neq \pi/2$. But if we set $\alpha = \pi/2$, equation (13) gives $\rho \equiv b \geqslant 1/\pi$. Further as $\alpha \to 0^+$ or $\alpha \to \pi^-$ we have

$$\lim \frac{\pi}{4}\,\frac{\sin\alpha}{\alpha(\pi-\alpha)} = \frac{1}{4}, \tag{14}$$

the Koebe constant. That $|a| \geqslant 1/4$ can be established by the same type of subordination argument, but it is interesting to observe that in this special case ($b = 0$) the universal functions are $F(z) = z/(1 \pm z)^2$, a rather abrupt change from $G(z)$ given by (2). We have proved

THEOREM 2. *Let D be the largest domain such that $D \subset f(E)$ for every $f(z)$ in* TR. *Then the upper boundary of D is given in polar coordinates (ρ,θ) by*

$$\rho = (\pi \sin \theta)/4\theta(\pi - \theta), \tag{15}$$

and the lower boundary is obtained by reflecting this curve in the real axis. The points $w = \pm 1/4$ are also part of the boundary of D.

3. **Comments.** M. S. Robertson [4] has proved that if $f(z) \in$ TR, then it has an integral representation

$$f(z) = \frac{1}{\pi}\int_0^{\pi} \frac{z}{1 - 2z\cos t + z^2}\, d\mu(t), \tag{16}$$

where $\mu(t)$ is nondecreasing on $[0, \pi]$ and

$$\int_0^{\pi} d\mu(t) = \pi.$$

Since $G(z)$ has such unusual properties it would be interesting to have $\mu(t)$ explicitly for this function. For one thing, we could compute the coefficients in the power series for $G(z)$—coefficients that seem to be difficult to compute directly from (2).

It is known [5] that if $f(z)$ is a normalized typically-real function, then

$$g(z) \equiv \left[(1 - z^2)/z\right] f(z) = 1 + p_1 z + p_2 z^2 + \dots$$

is in P, the set of normalized functions with $\operatorname{Re} g(z) > 0$ in E. One might ask for the largest domain covered by every function in this set. Here the corresponding question is trivial because for the set P, the set D that is always covered consists of the single point $w = 1$ (and does not form a domain). Even if we consider the subset of univalent functions in P, we get the same trivial answer: $D = \{1\}$.

The largest domain D covered by $f(E)$ for every $f(z)$ in a certain set M of normalized functions has been treated by other writers. Classical distortion theorems give the disk $|w| < 1/4$ if M is the set S of normalized univalent functions, and $|w| < 1/2$ for the subset of convex functions. Goodman and Saff [2] found D for the subset of S of functions convex in the direction of the imaginary axis. Further McGregor [3] has determined D for subsets of S for which all the coefficients are real and: (a) $f(z)$ is starlike, (b) $f(z)$ is convex, and (c) $f(z)$ is convex in the direction of the imaginary axis.

The largest domain covered by $f(E)$ for every $f(z)$ in M is sometimes referred to as the Koebe domain for the set M. For more references to papers on Koebe domains the reader should consult Bernardi's *Bibliography of schlicht functions*, Part I (1966) and Part II (1977), Courant Institute of Mathematical Sciences, New York University, New York.

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