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ISOPERIMETRIC INEQUALITIES INVOLVING HEAT FLOW UNDER LINEAR RADIATION CONDITIONS

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ABSTRACT. Under the assumption that a constant linear radiation condition holds on each boundary component, we show that the annulus and the strip are heat-flow minimizing with respect to area-preserving variations in one (for the annulus: the outer) boundary component.

1. **Introduction.** As in Figure 1, let Ω represent a simply connected region relative to $[0,1]\times R$ whose boundary relative to $[0,1]\times R$ consists of two simple arcs Γ^* and Γ. (Here, Γ^* and Γ have bounded curvature and are horizontal at their endpoints.) The heat flow crossing Ω is defined by $H = -\int_{\Gamma^*} \mathbf{D}_n u(p)\,|dp|$, where the temperature $u(p)$ is a harmonic function on Ω which satisfies the boundary conditions: $\mathbf{D}_n u = \lambda u$ on Γ, $\mathbf{D}_n u = \lambda^*(u-1)$ on Γ^*, and $\mathbf{D}_x u = 0$ on $(\{0,1\}\times R)\cap\Omega$. (Here λ and λ^* are positive constants and $\mathbf{D}_n u(p)$ at $p \in \Gamma^* \cup \Gamma$ is the normal derivative directed into Ω.)

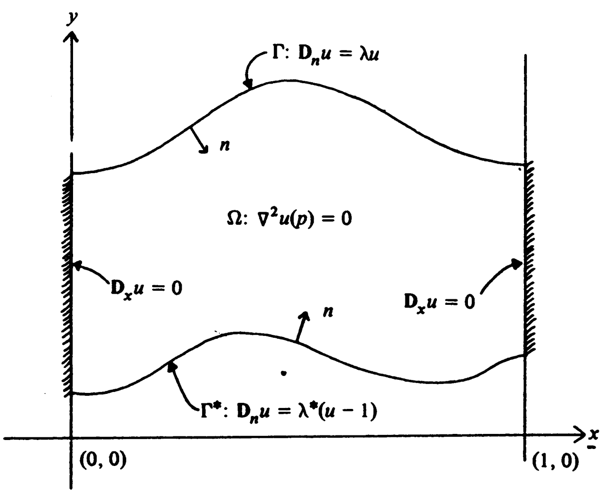

FIGURE 1. Definitions of Ω, Γ, Γ^* and $u(p)$.
We assume $\Gamma^* = [0, 1] \times \{0\}$ in §2.

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For any fixed $A > 0$, let $\{\Omega\}_A$ be the class of regions Ω such that $\Gamma^* = [0,1] \times \{0\}$, $|\Omega| \leqslant A$ (where $|\Omega|$ is the area of Ω), and the (inwardly directed) normal to Γ never points in the positive y-direction. In §2, we show that the region $[0,1] \times (0,A)$ is uniquely heat-flow minimizing in $\{\Omega\}_A$. It is intuitive that the horizontal strips $[0,1] \times (\alpha, \alpha + A)$, $\alpha \in R$, should be heat-flow minimizing in the class of all regions Ω having $|\Omega| \leqslant A$. This more general (probable) isoperimetric property of the strips $[0,1] \times (\alpha, \alpha + A)$ has not been proved. In the limiting case where $\lambda = \lambda^* = \infty$, the Newton radiation conditions reduce to the Dirichlet conditions $u = 1$ on Γ^* and $u = 0$ on Γ. In this case the general heat-flow-minimizing property of the strips $[0,1] \times (\alpha, \alpha + A)$ is equivalent to the main lemma in the proof of Grötzsch's Principle [3, pp. 140–142]. The analogous heat-flow-minimizing property of the annulus was proven by Carleman [2] and Szegö [7]. For the case where linear radiation conditions apply, a heat-flow-minimizing property of the annulus is discussed in §3.

2. **The main results.** Let $\overline{\Omega}$, $\overline{\Gamma}$, $\overline{\Gamma}^*$, $\bar{u}(p)$, and $\overline{H}$ refer to a second heat problem for which the assumptions in §1 hold. We assume throughout this section that $\Gamma^* = \overline{\Gamma}^* = [0,1] \times \{0\}$.

The temperature on $[0,1] \times (0,A)$ (for any $A > 0$) subject to our boundary conditions is

$$w_A(x,y) = w_A(y) = (\lambda^* + \lambda\lambda^*(A - y))(\lambda + \lambda^* + \lambda\lambda^* A)^{-1}. \tag{1}$$

The following theorem states the results to be proven in this section.

THEOREM 1. (a) *If* $\Omega \subset [0,1] \times (0,A)$ (*for some* $A > 0$), *then* $0 \leqslant u(x,y) \leqslant w_A(y)$ *for all* $(x,y) \in \Omega$.

(b) *If* $\overline{\Omega} \subset \Omega$, *then* $\overline{H} - H = \int_{\overline{\Gamma}} \bar{u}(p)(\lambda u(p) - \mathbf{D}_n u(p))\,|dp|$.

(c) *Assume* Γ *contains no point* p *at which the normal vector* $n(p)$ (*directed into* Ω) *points in the positive y-direction. Then*

$$H \geqslant \lambda\lambda^*(\lambda + \lambda^*)^{-1} - \lambda^2 \iint_\Omega w_y^2(y)\,dx\,dy.$$

Here, $w_y(y) = \lambda^*(\lambda + \lambda^* + \lambda\lambda^* y)^{-1}$, *as one obtains from equation* (1).

(d) *Let* $\{\Omega\}_A$ *be the class of regions such that* $|\Omega| \leqslant A$ *and the condition in part* (c) *is satisfied. Then* $[0,1] \times (0,A)$ *is uniquely heat-flow minimizing in* $\{\Omega\}_A$.

PROOF OF (a). Direct calculation shows that $|\nabla w_A(p)| \leqslant \lambda w_A(p)$ for all $p \in [0,1] \times [0,A]$. Therefore if $v(p) := w_A(p) - u(p)$, then $v(p)$ is a harmonic function on Ω which satisfies the boundary conditions: $\mathbf{D}_n v = \lambda^* v$ on Γ^*, $\mathbf{D}_n v \leqslant \lambda v$ on Γ, and $\mathbf{D}_x v = 0$ on $(\{0,1\} \times R) \cap \Omega$. Since $v(p)$ must achieve its minimum value at a point on $\Gamma^* \cup \Gamma$ (this follows from the usual maximum principle for the region ω defined in the Appendix) we conclude that $v(p) \geqslant 0$ in Ω.

PROOF OF (b). We have $\overline{H} - H = -\int_{\Gamma^*} \mathbf{D}_n v(p)|dp|$, where $v := \bar{u} - u$ on

$\bar{\Omega}$. v is harmonic in $\bar{\Omega}$ and satisfies the boundary conditions: $\mathbf{D}_n v = \lambda^* v$ on Γ^*, $\mathbf{D}_n v - \lambda v = \lambda u - \mathbf{D}_n u$ on $\bar{\Gamma}$, and $\mathbf{D}_x v = 0$ on $(\{0, 1\} \times R) \cap \bar{\Omega}$. Substitution of the boundary conditions on Γ^* yields

$$\bar{H} - H = -\lambda^* \int_{\Gamma^*} v = \int_{\Gamma^*} [v(\mathbf{D}_n \bar{u} - \lambda^* \bar{u}) - \bar{u}(\mathbf{D}_n v - \lambda^* v)]$$
$$= \int_{\Gamma^*} (v\mathbf{D}_n \bar{u} - \bar{u}\mathbf{D}_n v).$$

Then Green's second identity, applied to the functions v and $\bar{u}$ on $\bar{\Omega}$, yields

$$\bar{H} - H = \int_{\Gamma} (\bar{u}\mathbf{D}_n v - v\mathbf{D}_n \bar{u}) = \int_{\bar{\Gamma}} \left[\bar{u}(\mathbf{D}_n v - \lambda v) - v(\mathbf{D}_n \bar{u} - \lambda \bar{u}) \right]$$
$$= \int_{\bar{\Gamma}} \bar{u}(\lambda u - \mathbf{D}_n u).$$

The proof of part (d) (from part (c)). The inequality in (c) reduces to equality for any rectangle $[0, 1] \times (0, A)$, as can be shown by direct integration. (From equation (1): the heat flow across $[0, 1] \times (0, A)$ is $\lambda\lambda^*(\lambda + \lambda^* + \lambda\lambda^* A)^{-1}$.) Therefore, the proof that less heat flows across $[0, 1] \times (0, A)$ than any other admissible region Ω with $|\Omega| \leqslant A$ follows from the fact that $w_y(y)$ is a positive and strictly monotone decreasing function of y on $[0, \infty)$.

Proof of (c). An open disc with center p and radius ε is defined by $B_\varepsilon(p) = \{q \in [0,1] \times R \,|\, |q - p| < \varepsilon\}$. Choose a fixed $\varepsilon > 0$ so small that Ω equals the union of all open discs of radius ε contained in Ω. Let $\alpha^* = \max\{y | p = (x,y) \in \Gamma\}$. We define the monotone class of regions $\{\Omega_\alpha | 0 < \alpha \leqslant \alpha^*\}$ as follows. If $0 < \alpha \leqslant 2\varepsilon$, then $\Omega_\alpha = [0,1] \times (0,\alpha)$. If $2\varepsilon \leqslant \alpha \leqslant \alpha^*$, then Ω_α is the union of all open discs of radius ε which are contained in $\Omega \cap ([0,1] \times (0,\alpha))$. The class of regions $\{\Omega_\alpha | 0 < \alpha \leqslant \alpha^*\}$ thus defined has the following properties. (1) For each $\alpha \in (0,\alpha^*]$, the upper boundary Γ_α of Ω_α is a simple arc with bounded curvature which is horizontal at its endpoints. (2) For each α, the solution $u_\alpha(p)$ of our boundary value problem on Ω_α exists. Moreover, $\nabla u_\alpha(p)$ has a continuous extension to Closure(Ω_α) which is in fact Hölder continuous (exponent η) for any $0 < \eta < 1$. (3) For each α, the heat flow H_α crossing Ω_α is defined. H_α is a continuous function of α on $(0,\alpha^*]$. (Proofs for properties (2) and (3) are outlined in the appendix.) (4) $\Omega_\alpha \subset \Omega_\beta$ for $0 < \alpha \leqslant \beta \leqslant \alpha^*$, $\Omega_{\alpha^*} = \Omega$, and $\bigcup_{0<\alpha\leqslant\alpha^*} \Gamma_\alpha = \Omega \cup \Gamma$.

We now proceed to estimate the left-hand derivative $\mathbf{D}_\alpha^- H_\alpha = \operatorname{limit}_{\delta \to 0+} ((H_\alpha - H_{\alpha-\delta})/\delta)$ at each $\alpha \in (0,\alpha^*]$. For $0 < \alpha - \delta < \alpha \leqslant \alpha^*$, we have from Theorem 1(b) that

$$H_{\alpha-\delta} - H_\alpha = \int_{\Gamma_{\alpha-\delta}} u_{\alpha-\delta}(\lambda u_\alpha - \mathbf{D}_n u_\alpha). \tag{2}$$

It follows from the Hölder continuity of ∇u_α on Closure(Ω_α) (with exponent η) that $|\lambda u_\alpha - \mathbf{D}_n u_\alpha| = O(\delta^\eta)$ on $\Gamma_{\alpha-\delta}$. Therefore, $|u_{\alpha-\delta} - u_\alpha| = O(\delta^\eta)$ in $\Omega_{\alpha-\delta}$ by the maximum principle, and it follows that

(3) $$H_{\alpha-\delta} - H_\alpha = \int_{\Gamma_{\alpha-\delta}} u_\alpha(\lambda u_\alpha - \mathbf{D}_n u_\alpha) + O(\delta^{2\eta}).$$

The divergence theorem on the region $\omega_\delta := \Omega_\alpha \backslash \Omega_{\alpha-\delta}$ implies that

(4) $$\iint_{\omega_\delta} |\nabla u_\alpha|^2 = \int_{\Gamma_{\alpha-\delta}} u_\alpha \mathbf{D}_n u_\alpha - \int_{\Gamma_\alpha} u_\alpha \mathbf{D}_n u_\alpha.$$

(Here, the normal derivative $\mathbf{D}_n u_\alpha$ is directed into Ω_α on Γ_α and into $\Omega_{\alpha-\delta}$ on $\Gamma_{\alpha-\delta}$ as usual.) By substituting equation (4) into equation (3) and applying the condition: $\mathbf{D}_n u_\alpha = \lambda u_\alpha$ on Γ_α, one obtains

(5) $$H_{\alpha-\delta} - H_\alpha = \lambda\left[\int_{\Gamma_{\alpha-\delta}} u_\alpha^2 - \int_{\Gamma_\alpha} u_\alpha^2\right] - \iint_{\omega_\delta} |\nabla u_\alpha|^2 + O(\delta^{2\eta}).$$

However, if $\gamma_\alpha = \Gamma_\alpha \cap \Omega$, then

(6) $$\operatorname*{limit}_{\delta\to 0+} \frac{1}{\delta} \iint_{\omega_\delta} |\nabla u_\alpha|^2 = \int_{\gamma_\alpha} \phi |\nabla u_\alpha|^2,$$

where $\phi(p) = \text{limit}_{\delta\to 0+} (1/\delta)\cdot \text{Distance}(p, \Gamma_{\alpha-\delta}) \geqslant 0$ for each $p \in \gamma_\alpha$. Also,

(7) $$\operatorname*{limit}_{\delta\to 0+} \frac{\lambda}{\delta}\left[\int_{\Gamma_{\alpha-\delta}} u_\alpha^2 - \int_{\Gamma_\alpha} u_\alpha^2\right] = \lambda \int_{\gamma_\alpha} \phi(\mathbf{D}_n u_\alpha^2 - \kappa u_\alpha^2) = \int_{\gamma_\alpha} \phi(2\lambda^2 - \lambda\kappa) u_\alpha^2,$$

where $\kappa(p)$, which represents the curvature of γ_α at $p \in \gamma_\alpha$, is piecewise constant and assumes only the values 0 and $(1/\varepsilon)$. In the case where $\frac{1}{2} < \eta < 1$, equations (5), (6), and (7) show that $\mathbf{D}_\alpha^- H_\alpha$ exists for each $\alpha \in (0, \alpha^*]$. Since $|\nabla u_\alpha(p)| \geqslant \mathbf{D}_n u_\alpha(p) = \lambda u_\alpha(p)$ at each $p \in \gamma_\alpha$, we obtain from equations (5), (6), and (7) the estimate

(8) $$-\mathbf{D}_\alpha^- H_\alpha \leqslant \lambda^2 \int_{\gamma_\alpha} \phi u_\alpha^2, \qquad \alpha \in (0, \alpha^*].$$

From the expression for $w_\alpha(y)$ in equation (1) and the inequality in Theorem 1(a) we obtain (for $\alpha \in [\varepsilon, \alpha^*]$) the inequalities

(9) $$\begin{aligned} \mathbf{D}_\alpha^- \iint_{\Omega_\alpha} w_y^2(y)\,dx\,dy &= \int_{\gamma_\alpha} \phi(p) w_y^2(y)\,|dp| \geqslant w_\alpha^2(\alpha) \int_{\gamma_\alpha} \phi \\ &\geqslant w_\alpha^2(\alpha - \varepsilon) \int_{\gamma_\alpha} \phi - C\varepsilon \geqslant \int_{\gamma_\alpha} \phi u_\alpha^2 - C\varepsilon, \end{aligned}$$

where C can be chosen uniformly over all α and ε with $0 < \alpha - \varepsilon < \alpha \leqslant \alpha^*$. By combining equations (8) and (9), we obtain

(10) $$\mathbf{D}_\alpha^- F(\alpha) > 0 \quad \text{for } \varepsilon < \alpha \leqslant \alpha^*,$$

where

$$F(\alpha) = H_\alpha + \lambda^2 \iint_{\Omega_\alpha} w_y^2(y)\,dx\,dy + \lambda^2 C\varepsilon\alpha.$$

Since $\Omega_\alpha = [0, 1] \times (0, \alpha)$ for $\alpha \in (0, \varepsilon)$, it is easily shown that inequality (10) also holds in this case and that $\text{limit}_{\alpha\to 0+} H_\alpha = \lambda\lambda^*(\lambda + \lambda^*)^{-1}$. Since $F(\alpha)$ is continuous on $(0, \alpha^*]$, it follows essentially by the intermediate value theorem that

$$(11) \qquad F(\alpha^*) \geqslant \underset{\alpha\to 0+}{\text{limit}}\, F(\alpha) = \lambda\lambda^*(\lambda + \lambda^*)^{-1}.$$

Therefore

$$(12) \qquad H = H_{\alpha^*} \geqslant \lambda\lambda^*(\lambda + \lambda^*)^{-1} - \lambda^2 \iint_\Omega w_y^2(y)\,dx\,dy - \lambda^2 C\varepsilon\alpha^*.$$

The inequality asserted in part (c) now follows from the fact that $\varepsilon > 0$ can be chosen arbitrarily small.

3. **Doubly-connected regions in R^2.** In this section we assume Ω is a doubly-connected region whose inner and outer boundary components are, respectively, the unit circle Γ^* and a closed Jordan curve Γ having bounded curvature. The temperature $u(p)$ is a harmonic function in Ω satisfying the boundary conditions: $\mathbf{D}_n u = \lambda u$ on Γ and $\mathbf{D}_n u = \lambda^*(u - 1)$ on Γ^*. The rate of heat flow across Ω is $H = -\int_{\Gamma^*} \mathbf{D}_n u(p)|dp|$.

The solution of our temperature problem on the annulus $R(\alpha) = \{(r, \theta)| 1 < r < \alpha, 0 \leqslant \theta \leqslant 2\pi\}$ (for any $\alpha > 1$) is given by

$$(13) \quad w_\alpha(r,\theta) = w_\alpha(r) = (\lambda^* + \lambda^*\lambda\alpha\,\text{Ln}(\alpha/r))(\alpha\lambda + \lambda^*\lambda\alpha\,\text{Ln}(\alpha) + \lambda^*)^{-1}.$$

We summarize the results corresponding to Theorem 1 in the present context.

THEOREM 2. *Assume* $\lambda \geqslant 1$ *and* $\alpha > 1$. *Then*:

(a) *If* $\Omega \subset R(\alpha)$, *then* $0 \leqslant u(r,\theta) \leqslant w_\alpha(r)$ *for each* $(r,\theta) \in \Omega$.

(b) *If* Γ *contains no point* p *at which the normal vector* $n(p)$ (*directed into* Ω) *points in the outward radial direction, then*

$$H \geqslant 2\pi\lambda^*\lambda(\lambda^* + \lambda)^{-1} - \lambda \iint_\Omega \left(\lambda - \frac{1}{r}\right) w_r^2(r)\,r\,dr\,d\theta,$$

where $w_r(r) = \lambda^*(r\lambda + \lambda^*\lambda r\,\text{Ln}(r) + \lambda^*)^{-1}$.

(c) *Let* $\{\Omega\}_\alpha$ *be the class of regions such that* $|\Omega| \leqslant \pi(\alpha^2 - 1)$ *and the condition on* Γ *in part* (b) *is satisfied. Then the annulus* $R(\alpha)$ *is uniquely heat-flow minimizing in* $\{\Omega\}_\alpha$.

The proofs are analogous to those in §2. (Clearly Theorem 1(b) carries over without change.)

4. **Remarks concerning a more general problem.** Let $\{\Omega\}_A$ be the class of doubly-connected regions Ω with $|\Omega| \leqslant A$, whose inner boundary component is a fixed closed curve Γ^*. If a region Ω is heat-flow minimizing (subject to our radiation conditions) in $\{\Omega\}_A$ and has a sufficiently smooth outer boundary Γ,

then a variational procedure shows that the quantity $u(p)(\lambda^2 u(p) - \mathbf{D}_n^2 u(p))$ is a constant along Γ. That a boundary Γ satisfying this condition actually exists and is heat-flow minimizing has not been proven. (In the limiting case where $\lambda^* = \lambda = \infty$, the necessary condition reduces to: $\mathbf{D}_n u$ constant on Γ. In this case, the heat-flow-minimizing boundary Γ exists provided that Γ^* bounds a starlike region. See Acker [1].) This paper has handled the case where $\lambda^* > 0$, $\lambda \geqslant 1$, and Γ^* is the unit circle.

5. Appendix.

LEMMA. *In the context of* §2:

(a) $\nabla u(p)$ *has a continuous extension to* Closure(Ω). *In fact* $\nabla u(p)$ *is Hölder continuous on* Closure(Ω) *for any Hölder exponent* $0 < \eta < 1$.

(b) H_α *is a continuous function of* α *on* $(0, \alpha^*]$.

PROOF OF PART (a). The region $\Omega_e = \{(x,y) | (x,y) \in \Omega \text{ or } (-x,y) \in \Omega\}$ maps under $F(z) = \exp(-i\pi z)$ into a doubly-connected region ω whose inner and outer boundary components γ^* and γ have bounded curvature. The solution $u(p)$ (extended to Ω_e by $u(x,y) = u(-x,y)$) maps under $F(z)$ into a harmonic function $v(p)$ in ω which satisfies the boundary condition

$$\mathbf{D}_n v(p) = \lambda(p) v(p) + f(p) \quad \text{on } \partial\omega = \gamma^* \cup \gamma.$$

Here, $\lambda(p) = (\lambda/\pi|p|)$, $f(p) = \lambda$ on γ^*, and $f(p) = 0$ on γ. Now $v(p)$ can be expressed in the form

$$v(p) = \int_{\partial\omega} \psi(q) \operatorname{Ln}(|q-p|)\,|dq|, \tag{14}$$

where $\psi(p)$ (defined on $\partial\omega$) is a solution in $H(1)$ (i.e., Hölder continuous for any exponent $0 < \eta < 1$) of the integral equation

$$\pi\psi(p) - \int_{\partial\omega} \psi(q)\left[\frac{(q-p)\cdot n(p)}{|q-p|^2} + \lambda(p) \operatorname{Ln}(|q-p|)\right]|dq| = f(p), \qquad p \in \partial\omega. \tag{15}$$

(See [6, XVIII, §3].) One can show using well-known integral formulas for

$$\frac{\partial v(p)}{\partial n(p)} = \operatorname*{limit}_{q \to p} \frac{\partial v(q)}{\partial n(p)} \quad \text{and} \quad \frac{\partial v(p)}{\partial T(p)} = \operatorname*{limit}_{q \to p} \frac{\partial v(q)}{\partial T(p)}$$

(where $p \in \partial\omega$, $q \in \omega$, and $T(p)$ is a unit tangent to $\partial\omega$ at p) that these functions are in $H(1)$ on $\partial\omega$. (See [5, 2, §13] and [4, §29].) Thus, the function $G(z) = \mathbf{D}_x v(z) - i\mathbf{D}_y v(z)$ (analytic in ω) has a continuous extension to $\partial\omega$ which is in $H(1)$ on $\partial\omega$. Thus, $G(z)$ (or $\nabla v(p)$) is in $H(1)$ on $\omega \cup \partial\omega$ by [5, 2, §22].

PROOF OF PART (b). For each $0 < \beta \leqslant \alpha^*$, let ω_β correspond to Ω_β as in part (a), and let equation (15β) refer to equation (15) when $\omega = \omega_\beta$. If $\beta \to \alpha$, one can show by considering equation (15β) as a perturbation of equation (15α) that $u_\beta(p) \to u_\alpha(p)$ (and therefore that $\nabla u_\beta(p) \to \nabla u_\alpha(p)$) uniformly on any compact subset of Ω_α. The result follows from this.

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