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# A PROBLEM ON NOETHERIAN LOCAL RINGS OF CHARACTERISTIC $p$

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ABSTRACT. Let $(A, m, k)$ be a one-dimensional Noetherian local ring of characteristic $p$ ($p > 0$, a prime number) and assume that the Frobenius endomorphism $F$ of $A$ is finite. Further assume that the field $k$ is algebraically closed and that it is contained in $A$. Let $B$ denote $A$ when it is regarded as an $A$-algebra by $F$. Then, if $\operatorname{Hom}_A(B, A) \cong B$ as $B$-modules, $A$ is a Macaulay local ring and $r(A) \equiv \dim_k \operatorname{Ext}_A^1(k, A) \leqslant \max\{\sharp \operatorname{Ass} \hat{A} - 1, 1\}$ where $\hat{A}$ denotes the $m$-adic completion of $A$. Thus, in case $\sharp \operatorname{Ass} \hat{A} \leqslant 2$, $A$ is a Gorenstein local ring if and only if $\operatorname{Hom}_A(B, A) \cong B$ as $B$-modules. If $\sharp \operatorname{Ass} \hat{A} \geqslant 3$ this assertion is not true and the counterexamples are given.

1. **Introduction.** Let $(A, m, k)$ be a Noetherian local ring of characteristic $p$ ($p > 0$, a prime number) and assume that the Frobenius endomorphism $F$ of $A$ is finite. (For example, $A$ is essentially of finite type over a perfect field of positive characteristic or its completion.) We denote $A$ by $B$ if we regard $A$ as an $A$-algebra by $F$. The purpose of this note is to answer the question [5]: If $A$ is a Macaulay ring and if $\operatorname{Hom}_A(B, A) \cong B$ as $B$-modules, is $A$ a Gorenstein ring? Note that the converse of this question is always true. Our main results are contained in the following two theorems.

THEOREM (1.1). *The following two conditions are equivalent.*

(1) *$A$ is a Gorenstein ring.*

(2) $\operatorname{Hom}_A(B, A) \cong B$ *as $B$-modules and* $\operatorname{Ext}_A^i(B, A) = (0)$ *for* $0 < i \leqslant \operatorname{depth} A$.

THEOREM (1.2). *Suppose that $A$ contains $k$ and assume that $k$ is algebraically closed. If* $\dim A = 1$ *and* $\operatorname{Hom}_A(B, A) \cong B$ *as $B$-modules, then $A$ is a Macaulay ring and* $r(A) \equiv \dim_k \operatorname{Ext}_A^1(k, A) \leqslant \sharp \operatorname{Ass} \hat{A}$. *Moreover, if* $\sharp \operatorname{Ass} \hat{A} \geqslant 2$, *then this inequality is strict.*

Here $\hat{A}$ denotes the $m$-adic completion of $A$. Recall that $A$ is a Gorenstein ring if and only if $r(A) = 1$.

COROLLARY (1.3). *Suppose that $A$ contains $k$ and assume $k$ to be algebraically closed. Suppose that* $\dim A = 1$ *and* $\sharp \operatorname{Ass} \hat{A} \leqslant 2$. *Then $A$ is a Gorenstein ring if and only if* $\operatorname{Hom}_A(B, A) \cong B$ *as $B$-modules.*

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This gives a best possible generalization of Korollar 5.12 of [5]. In fact we will give by Example (2.8) that, for every integer $n \geqslant 2$ and for every perfect field $k$ of positive characteristic, there is a one-dimensional Noetherian local ring $A$ which satisfies the following conditions: (1) The Frobenius endomorphism $F$ of $A$ is finite. (2) $A$ contains $k$ and $k$ coincides with the residue field of $A$. (3) $\mathrm{Hom}_A(B, A) \cong B$ as $B$-modules. (4) $r(A) = n - 1$. (5) $\sharp \operatorname{Ass} \hat{A} = n$. Of course, for $n \geqslant 3$, this ring $A$ is a counterexample of our question.

2. **Proof of the theorems.** Let $M$ be an $A$-module. We denote $M$ by ${}_BM$ if we regard $M$ as a $B$-module.

LEMMA (2.1). *Let $E$ be an injective $A$-module. Then* $\mathrm{Hom}_A(B, E) \cong {}_BE$ *as $B$-modules.*

PROOF. We may assume that $E = E_A(A/p)$ (the injective envelope of $A/p$) for some $p \in \operatorname{Spec} A$. Since $\mathrm{Hom}_A(B, E)$ is an injective $B$-module with $\operatorname{Ass}_B \mathrm{Hom}_A(B, E) = \{q\}$ ($q$ denotes $p$ as an ideal of $B$), we can express $\mathrm{Hom}_A(B, E) = E_B(B/q)^{(n)}$ for some integer $n > 0$. On the other hand, as

$$A_p \otimes_A \mathrm{Hom}_A(B, E) = \mathrm{Hom}_{A_p}(B_q, E_{A_p}(A_p/pA_p))$$

and as the latter is isomorphic to $E_{B_q}(B_q/qB_q)$, we conclude that $n = 1$.

LEMMA (2.2) [5, KOROLLAR 4.3]. *Let $M$ be an Artinian $A$-module. If* $\mathrm{Hom}_A(B, M) \cong {}_BM$ *as $B$-modules, then $M$ is an injective $A$-module.*

For every $A$-module $M$ and for every integer $i \geqslant 0$, we put

$$H_m^i(M) = \varinjlim_t \mathrm{Ext}_A^i(A/m^t, M)$$

and call it the $i$th local cohomology module of $M$ [4]. $H_m^0(\ )$ is a left exact functor and $\{H_m^i(\ )\}_{i \geqslant 0}$ are as its derived functors.

LEMMA (2.3) [5, LEMMA 5.1]. *Let $n$ denote the maximal ideal of $B$. Then* $\mathrm{Hom}_A(B, H_m^0(\ )) = H_n^0(\mathrm{Hom}_A(B, \ ))$.

PROOF OF THEOREM (1.1). That (1) implies (2) is found in [7] (cf. 5.19 and Korollar 6.8). To show that (2) implies (1) we have only to prove that $H_m^s(A)$ is an injective $A$-module where $s = \operatorname{depth} A$ by Corollary 3.9 of [2]. Let the following sequence

$$0 \to A \xrightarrow{d^{-1}} E^0 \to \cdots \to E^i \xrightarrow{d^i} E^{i+1} \to \cdots \tag{1}$$

be the minimal injective resolution of $A$. Applying $\mathrm{Hom}_A(B, \ )$ to (1), we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_A(B, A) &\xrightarrow{(d^{-1})_*} \mathrm{Hom}_A(B, E^0) \to \cdots \\ &\to \mathrm{Hom}_A(B, E^s) \xrightarrow{(d^s)_*} \mathrm{Hom}_A(B, E^{s+1}) \end{aligned} \tag{2}$$

of $B$-modules. Therefore, recalling that $\operatorname{Hom}_A(B,\ )$ preserves essential monomorphisms (see e.g. [6, Proposition 3]), we have that $(d^i)_*$ is an essential $B$ linear map for all $i = -1, 0, \ldots, s-1$. Hence, by (2.1), the first $s$ terms of (2) coincide with the first $s$ terms of the minimal injective resolution of $\operatorname{Hom}_A(B, A) = B$. Thus there is a family $\{f^i\}_{i=-1,0,\ldots,s}$ of $B$-isomorphisms such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 \longrightarrow & B & \longrightarrow & {}_BE^0 & \longrightarrow \cdots \longrightarrow & {}_BE^s & \xrightarrow{{}_Bd^s} & {}_BE^{s+1} \\ & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow f^s & & \\ 0 \longrightarrow & \operatorname{Hom}_A(B, A) & \longrightarrow & \operatorname{Hom}_A(B, E^0) & \longrightarrow \cdots \longrightarrow & \operatorname{Hom}_A(B, E^s) & \xrightarrow{(d^s)_*} & \operatorname{Hom}_A(B, E^{s+1}). \end{array}$$

Moreover, as ${}_Bd^s$ is an essential $B$-linear map, we can find a $B$-monomorphism $f^{s+1}$ so that the following square

$$\begin{array}{ccc} {}_BE^s & \xrightarrow{{}_Bd^s} & {}_BE^{s+1} \\ \downarrow\wr f^s & & \downarrow f^{s+1} \\ \operatorname{Hom}_A(B, E^s) & \xrightarrow{(d^s)_*} & \operatorname{Hom}_A(B, E^{s+1}) \end{array} \tag{3}$$

is commutative. Note that, since $\operatorname{Hom}_A(B, E^{s+1}) \cong {}_BE^{s+1}$ by (2.1), then $f^{s+1}$ is actually a $B$-isomorphism. Applying $H_n^0(\ )$ to (3) we have a commutative square by (2.3):

$$\begin{array}{ccc} {}_BH_m^0(E^s) & \xrightarrow{{}_BH_m^0(d^s)} & {}_BH_m^0(E^{s+1}) \\ \downarrow\wr & & \downarrow\wr \\ \operatorname{Hom}_A(B, H_m^0(E^s)) & \xrightarrow{(H_m^0(d^s))_*} & \operatorname{Hom}_A(B, H_m^0(E^{s+1})). \end{array}$$

Thus we have ${}_BH_m^s(A) \cong \operatorname{Hom}_A(B, H_m^s(A))$, since $H_m^s(A) = \operatorname{Ker} H_m^0(d^s)$. This yields by (2.2) that $H_m^s(A)$ is an injective $A$-module.

COROLLARY (2.4). *Suppose that* $\operatorname{Hom}_A(B, A) \cong B$ *as B-modules. Then the total quotient ring of A is a Gorenstein ring. If* $\operatorname{depth} A = 0$, *then A is an Artinian Gorenstein local ring.*

COROLLARY (2.5). *Suppose that A is a UFD. Then*

(1) $\hat{A}_p$ *is a Gorenstein ring for every prime ideal p of* $\hat{A}$ *such that* $\operatorname{depth} \hat{A}_p \leqslant 1$. *Thus* $\hat{A}$ *satisfies the condition* $(S_2)$.

(2) *The Gorenstein locus V of A is open in* $\operatorname{Spec} A$. *In fact, if* $\dim A = d$, *then* $V = \operatorname{Spec} A - \bigcup_{i=1}^d \operatorname{Supp}_A \operatorname{Ext}_A^i(B, A)$.

(3) *A is a Gorenstein ring if and only if* $\operatorname{Ext}_A^i(B, A) = (0)$ *for every* $0 < i \leqslant \operatorname{depth} A$.

PROOF. By Korollar 5.15 of [5] we know that $\mathrm{Hom}_A(B,A) \cong B$ as $B$-modules. Hence (2) and (3) are obvious by (1.1). For (1), let $p$ be a prime ideal of $A$ with $\dim A_p = 1$. Then $A_p \otimes_A \mathrm{Ext}^1_A(B,A) = \mathrm{Ext}^1_{A_p}(B_q, A_p) = (0)$ ($q$ denotes $p$ as an ideal of $B$) by (1.1), since $A_p$ is a $DVR$. Thus we see that $\mathrm{grade}_A \mathrm{Ext}^1_A(B,A) \geqslant 2$. Therefore $\mathrm{grade}_{\hat{A}} \mathrm{Ext}^1_{\hat{A}}(\hat{B},\hat{A}) \geqslant 2$ and hence, again by (1.1), the ring $\hat{A}_p$ is a Gorenstein ring for every prime ideal $p$ of $\hat{A}$ such that $\mathrm{depth}\, \hat{A}_p \leqslant 1$.

Suppose that $A$ is a Macaulay ring and let $K$ be an $A$-module. $K$ is said to be a canonical module of $A$ if $\hat{A} \otimes_A K \cup \mathrm{Hom}_{\hat{A}}(H^d_m(\hat{A}), E_{\hat{A}}(k))$ ($d = \dim A$) as $\hat{A}$-modules [7]. The canonical module is uniquely determined up to isomorphisms for $A$ if it exists. A canonical module is called a canonical ideal if it is contained in $A$ as an ideal. It is known that $A$ has a canonical module $K_A$ if and only if $A$ is a homomorphic image of a Gorenstein local ring $R$ (cf. [9]). In this case $K_A \cong \mathrm{Ext}^t_R(A,R)$ where $t = \dim R - \dim A$, and $A$ is a Gorenstein ring if and only if $K_A \cong A$ [7], [10]. Various properties of canonical modules are discussed by [7] to which we shall often refer in this note.

LEMMA (2.6). *Suppose that* $\dim A = 1$. *Then the following three conditions are equivalent.*

(1) $\mathrm{Hom}_A(B,A) \cong B$ *as B-modules.*

(2) *A is a Macaulay ring and the total quotient ring of A is a Gorenstein ring. The ring A has a canonical ideal and, for every canonical ideal K of A, the ideal* $K^{(p)}$ *is again a canonical ideal of A. (Here* $K^{(p)}$ *denotes the ideal of A generated by the elements of the form* $x^p$ *with* $x \in K$.)

(2′) *A is a Macaulay ring and the total quotient ring of A is a Gorenstein ring. The ring A has a canonical ideal K such that the ideal* $K^{(p)}$ *is again a canonical ideal of A.*

PROOF. Suppose (1). By (2.4) we see that $\mathrm{depth}\, A > 0$ and hence $A$ is a Macaulay ring. Moreover since $\mathrm{Hom}_{\hat{A}}(\hat{B},\hat{A}) \cong \hat{B}$ as $\hat{B}$-modules, the total quotient ring of $\hat{A}$ is a Gorenstein ring by (2.4). Thus the total quotient ring of $A$ is a Gorenstein ring and the ring $A$ has a canonical ideal (cf. [3, Theorem] or [7, Satz 6.21]). Now let $K$ be a canonical ideal of $A$. Then ${}_BK \cong \mathrm{Hom}_A(B,K)$ by 5.19 of [7]. Hence, by the assumption, there is an isomorphism ${}_BK \cong \mathrm{Hom}_A(\mathrm{Hom}_A(B,A),K)$. On the other hand, we have $\mathrm{Hom}_A(B,A) \cong \mathrm{Hom}_A(KB,K)$ by Korollar 5.9 of [5]. Thus ${}_BK \cong \mathrm{Hom}_A(\mathrm{Hom}_A(KB,K),K)$. Since $KB$ is a Macaulay $A$-module of $\dim_A KB = 1$, we have by Satz 6.1 of [7] that $\mathrm{Hom}_A(\mathrm{Hom}_A(KB,K),K) \cong KB$. Thus ${}_BK \cong KB$ as $B$-modules and so the ideal $K^{(p)}$ is a canonical ideal of $A$, and we have verified the statements in (2). Obviously (2) implies (2′).

Suppose (2′). By Korollar 5.9 of [5], we know that $\mathrm{Hom}_A(B,A) \cong \mathrm{Hom}_A(KB,K)$. Since $KB \cong {}_BK$ by the assumption, then $\mathrm{Hom}_A(KB,K) \cong \mathrm{Hom}_A(\mathrm{Hom}_A(B,K),K)$. (Recall that ${}_BK \cong \mathrm{Hom}_A(B,K)$ by 5.19 of [7].)

Thus, since $\mathrm{Hom}_A(\mathrm{Hom}_A(B,K),K) \cong B$ by Satz 6.1 of [7], we conclude that $\mathrm{Hom}_A(B,A) \cong B$ as desired.

REMARK (2.7). The conditions of (2.6) are nontrivial. For example, let $k$ be a perfect field of characteristic 2 and put $A = k[|x,y,z|]$ with relation $x^2 = xy = yz = 0$. Then $\mathrm{Ass}\,A = \{(x,y),(x,z)\}$ and $r(A) = 2$. $K = (x+y, z)$ is a canonical ideal of $A$ and $K \not\cong K^{(2)}$. Thus $\mathrm{Hom}_A(B,A) \not\cong B$ as $B$-modules in this case.

PROOF OF THEOREM (1.2). We may assume that $A$ is complete. We denote by $Q$ the total quotient ring of $A$ and let $K$ be a canonical ideal of $A$. Then $K^{(p)} = aK$ for some invertible element $a$ of $Q$, since $K^{(p)}$ is again a canonical ideal of $A$ by (2.6) (cf. [7, Satz 2.8]). Now let us define a map $f\colon Q \to Q$ by $f(x) = x^p/a$. Then $f$ is a so-called $p$-linear endomorphism of $Q$ (cf. Hartshorne-Speiser [8]). In the following we will preserve the definitions and notations of §1 of [8]. Since $K$ is an $(A,F)$-submodule of $Q$, the module $Q/K$ becomes an $(A,F)$-module so that the sequence

$$0 \to K \to Q \to Q/K \to 0$$

is an exact sequence of $(A,F)$-modules. Therefore, by Theorem 1.13 of [8], we have an exact sequence

$$(*) \qquad 0 \to K_s \to Q_s \to (Q/K)_s \to 0$$

of finite-dimensional level $(k,F)$-modules, since $Q/K$ is an Artinian $A$-module. Similarly, applying the functor $[\ ]_s$ to the exact sequence $0 \to mK \to K \to K/mK \to 0$, we have an exact sequence

$$0 \to (mK)_s \to K_s \to (K/mK)_s \to 0.$$

Recalling $(mK)_s = (0)$ (cf. [8, Proof of Lemma 1.16]), we conclude that $K_s = (K/mK)_s$. On the other hand, since $K$ is generated by $\mathrm{Im} f_K$ as an $A$-module, then $f_{K/mK}$ is a surjective map. (Note that $k$ is algebraically closed.) Thus $(K/mK)_s = K/mK$ and hence $\dim_k K_s = r(A)$, since $r(A)$ is equal to the number of minimal generators of $K$ (cf. [7, Satz 6.10]). Therefore by $(*)$, to prove the first statement of the theorem, it suffices to show that $\dim_k Q_s \leqslant \#\mathrm{Ass}\,A$. We put $n = \#\mathrm{Ass}\,A$.

Let us express $Q = \prod_{i=1}^n A_i$ as a direct product of Artinian local rings $\{(A_i, m_i)\}_{1 \leqslant i \leqslant n}$ and let $a = a_1 + a_2 + \cdots + a_n$ be the corresponding decomposition. Then $a_i$ is a unit of $A_i$ and, if we define a map $f_i\colon A_i \to A_i$ by $f_i(x) = x^p/a_i$, then $(A_i, f_i)$ is an $(A,F)$-module and $Q = \bigoplus_{i=1}^n A_i$ as $(A,F)$-modules. Moreover, since $Q_s$ is a finite-dimensional level $(k,F)$-module, so is $(A_i)_s$. Now put $d_i = \dim_k (A_i)_s$. Of course $\dim_k Q_s = \sum_{i=1}^n d_i$. In the following we will show that $d_i \leqslant 1$.

Assume that $d_i \neq 0$. Then, by a well-known lemma (see [1, p. 233]), there is a $k$-basis $\langle e_1, e_2, \ldots, e_{d_i}\rangle$ of $(A_i)_s$ such that

$$(**) \qquad e_j^p = a_i e_j \qquad (j = 1, 2, \ldots, d_i).$$

Note that every $e_j$ is a unit of $A_i$. (If not, $e_j$ is nilpotent and therefore, by (**), $e_j = 0$.) Hence we have $a_i = e_j^{p-1} = e_h^{p-1} (j, h = 1, 2, \dots, d_i)$. Thus, for every $j$ and $h$, we can find $u_{jh} \in k$ so that $u_{jh}^{p-1} = 1$ and $u_{jh} - e_j/e_h \in m_i$. But as $u_{jh} - e_j/e_h = (u_{jh} - e_j/e_h)^p$, we conclude that $u_{jh} = e_j/e_h$ and therefore $e_j \in ke_h$. Thus $d_i = 1$ and we complete the proof of the first statement.

Now consider the second one. We put $p_i = m_i \cap A$ for every $1 \leqslant i \leqslant n$ and assume that $r(A) = n$ ($\geqslant 2$). Then we have that $K_s = Q_s$ $(= \bigoplus_{i=1}^n (A_i)_s)$ and that $(A_i)_s \neq (0)$ for every $1 \leqslant i \leqslant n$, since $\dim_k K_s = \dim_k Q_s = n$ by the assumption. Let $\{x_i\}_{1 \leqslant i \leqslant n}$ be a family of elements of $K_s$ such that $0 \neq x_i \in A_i$. Then we have that $x_i x_j = 0$ if $i \neq j$. Moreover $x_i \in p_j$ if $i \neq j$, since $A_h = A_{p_h}$ for every $1 \leqslant h \leqslant n$. Of course $\{x_1, x_2, \dots, x_n\}$ are linearly independent over $k$ and therefore $K_s = \sum_{i=1}^n kx_i$. Thus we have that $K = (x_1, x_2, \dots, x_n)$ by Nakayama's lemma, since $K_s = K/mK$. In the following we will show that $K = (x_1) \oplus (x_2, \dots, x_n)$.

We put $x = x_1 + x_2 + \cdots + x_n$. Then $x \notin \bigcup_{i=1}^n p_i$. (If not, $x_i \in p_i$ for some $i$ and hence $K = (x_1, x_2, \dots, x_n) \subset p_i$. But this is impossible since the ideal $K$ contains a non-zerodivisor of $A$.) Therefore $x$ is a non-zerodivisor of $A$. Let $y$ be an element of the ideal $(x_1) \cap (x_2, \dots, x_n)$ and express $y = c_1 x_1 = c_2 x_2 + \cdots + c_n x_n$ for some $c_i \in A$. Then

$$
\begin{aligned}
yx &= (c_1 x_1)(x_1 + x_2 + \cdots + x_n) \\
&= (c_1 x_1) x_1 \\
&= (c_2 x_2 + \cdots + c_n x_n) x_1 = 0,
\end{aligned}
$$

and so $y = 0$ since $x$ is a non-zerodivisor of $A$. Thus we have that $K = (x_1) \oplus (x_2, \dots, x_n)$. But this is impossible, since $K$ is indecomposable as an $A$-module. (Cf. [10]. The terminology "a Gorenstein module of rank one" is another name of the canonical module.) Hence we conclude that $r(A) < n$ and we have verified the statements in the theorem.

EXAMPLE (2.8). Let $n \geqslant 2$ be an integer and let $k$ be a perfect field of positive characteristic $p$. Let $P = k[|X_1, X_2, \dots, X_n|]$ be a formal power series ring and let $a$ denote the ideal $(X_i X_j / 1 \leqslant i < j \leqslant n)$ of $P$. We denote $P/a$ by $A$ and put $x_i = X_i \bmod a$. Then we have

(1) The Frobenius endomorphism $F$ of $A$ is finite.

(2) $a = \bigcap_{i=1}^n q_i$ where $q_i = (X_1, \dots, \hat{X}_i, \dots, X_n)$. Hence the ring $A$ is reduced with $\dim A = 1$ and $\sharp \operatorname{Ass} A = n$.

(3) $K = (x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n)$ is a canonical ideal of $A$ and $r(A) = n - 1$.

(4) Let $x = x_1 + x_2 + \cdots + x_n$. Then $x$ is a non-zerodivisor of $A$ and the map $f\colon K \to K^{(p)}$, $f(y) = x^{p-1} y$, is an $A$-isomorphism. Hence $\operatorname{Hom}_A(B, A) \cong B$ as $B$-modules in this case (cf. (2.6)).

Of course, for $n \geqslant 3$, the ring $A$ is a counterexample of our question.

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