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A NOTE ON GOOD REDUCTION OF SIMPLE ABELIAN VARIETIES[1]

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ABSTRACT. In this note it is shown that the reduction of a simple abelian variety of dimension > 2, defined over an algebraic number field, at any finite good prime need not be simple. We give an example of a two-dimensional simple abelian variety defined over an algebraic number field whose reduction at any finite prime is isogenous either to a product of ordinary elliptic curves or to a product of supersingular elliptic curves.

It is well known [7] that an abelian variety A defined over an algebraic number field K has good reduction almost everywhere. In other words, if O is the ring of integers in K, Σ the set of all finite primes of K, and S a finite subset of Σ, then for every prime p in $\Sigma - S$, there exists an abelian scheme X over O_p such that $X \times_{O_p} K \simeq A$. Serre and Tate [6] have also proved that if A has sufficiently many complex multiplications, then it has potentially good reduction everywhere. It is clear that if the reduction $\tilde{A}$ of an abelian variety A is simple, then A itself is simple, because $\mathrm{End}^0(A) \subseteq \mathrm{End}^0(\tilde{A})$ and the latter is a division algebra over $\mathbf{Q}$. One can ask whether the converse is true. More precisely, if A is simple, then is $\tilde{A}$ also simple? In this note we shall see that this need not be the case.

Notations and terminology. We employ the following notation:
$\mathbf{Z} =$ the ring of rational integers; $\mathbf{Q} =$ the field of rationals;
$\mathbf{Z}_p =$ the ring of p-adic integers; $\mathbf{Q}_p =$ the field of fractions of $\mathbf{Z}_p$;
$\overline{\mathbf{Q}}_p =$ the algebraic closure of $\mathbf{Q}_p$; $\mathbf{C} =$ the field of complex numbers.
The principal reference for abelian varieties is [4]. If A is an abelian variety defined over a field K, let $\mathrm{End}^0(A) = \mathrm{End}_K(A) \otimes \mathbf{Q}$. If the characteristic of K is $p > 0$, let $T_p(A)$ denote the Barsotti-Tate group (p-divisible group in the terminology of Serre and Tate) associated to A. It has height $2g$, where g is the dimension of A. $T_p(A)_{\mathrm{red}} =$ the étale part of $T_p(A)$. Let M denote the Dieudonné module of $T_p(A)$. If K is a perfect field, then M is a module over $W(K)[F, V]$, where $W(K)$ is the ring of infinite Witt vectors over K and F, V

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are indeterminates such that (i) $FV = VF = p$ and (ii) $F\alpha = \alpha^{\sigma}F$, $\alpha V = V\alpha^{\sigma}$ for $\alpha \in W(K)$; here σ is the unique automorphism of $W(K)$ inducing the map $x \to x^p$ in K. The module M is free of rank $2g$ over $W(K)$. For details on Dieudonné modules, see [5]. The integer $p(A)$ will denote the p-rank of an abelian variety A.

THEOREM. *Let A be an abelian variety of dimension $\geqslant 2$, defined over a field k of positive characteristic p. Let $p(A) = 1$. Then* $\mathrm{End}^0(A)$ *will never contain a simple subalgebra which is not a field.*

PROOF. Suppose that $\mathrm{End}^0(A)$ contains a simple subalgebra L such that $1_A = 1_L$. Let C be the center of L. Then from the general theory of simple algebras, we have $[L:C] = d^2$ and $[C:\mathbf{Q}] = e$ for some natural numbers d and e. Let ρ be the representation of $L \otimes \mathbf{Q}_p$ on the extended Dieudonné module $M \otimes \mathbf{Q}_p$ of $T_p(A)$. This representation is induced from the representation of $\mathrm{End}^0(A)$. It has degree $2g$ over $W(k) \otimes \mathbf{Q}_p$. The representation ρ splits into three parts corresponding to the splitting of M into $M^{\text{ét,ét}} \oplus M^{\text{ét,loc}} \oplus M^{\text{loc,loc}}$, where $M^{\text{ét,ét}}$ is the étale part of M whose Cartier dual is also étale, etc. Write $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$. The first representation ρ_1 is the representation over $\mathbf{Q}_p$ of $L \otimes \mathbf{Q}_p$ on the p-adic Tate module $T_p(A)_{\text{red}} \otimes \overline{\mathbf{Q}}_p$. Since $p(A) = 1$, $T_p(A) \otimes \overline{\mathbf{Q}}_p$ is isomorphic to $\overline{\mathbf{Q}}_p$. Decompose $L \otimes \overline{\mathbf{Q}}_p$ into a product $\prod_1^e M_d(\overline{\mathbf{Q}}_p)$, where $M_d(\overline{\mathbf{Q}}_p)$ is the algebra of matrices of degree d over $\overline{\mathbf{Q}}_p$. Since the identity of L is represented by the identity matrix, the representation ρ_1 does not contain the zero representation. Consequently $d = 1$; that is L is a field. Q.E.D.

COROLLARY 1. *Let M_2 be the one-dimensional moduli scheme of two-dimensional abelian varieties defined over* $\mathbf{C}$ *whose endomorphism ring is an order in an indefinite quaternion division algebra H over* $\mathbf{Q}$. *Then the fiber at any closed point is a simple abelian variety over an algebraic number field K whose reduction at every finite prime of K is either ordinary or has p-rank* 0.

REMARK. The abelian varieties corresponding to closed points as in Corollary 1 are called *false elliptic curves* by J.-P. Serre for the reason that they behave like elliptic curves when we reduce at a finite prime. Recall that an elliptic curve in positive characteristic is either ordinary or supersingular, i.e. the p-rank is either one or zero.

PROOF OF COROLLARY 1. As Shimura has observed in [8], M_2 has a canonical nonsingular model defined over $\mathbf{Q}$ which we again denote by M_2. Let A be a two-dimensional abelian variety corresponding to any closed point of M_2. A is defined over a number field, say, K. It has potential good reduction at any finite prime of K. For example, see [3]. By extending K to a finite extension if necessary, we can assume that A has good reduction everywhere. For any finite prime p of K, let $\tilde{A}^p$ be the reduction of A at p. $\tilde{A}^p$ is defined over a finite field of characteristic p. Since $\mathrm{End}^0(A) \subseteq \mathrm{End}^0(\tilde{A}^p)$, $\mathrm{End}^0(\tilde{A}^p)$ contains an indefinite quaternion division algebra and so by the theorem, $\tilde{A}^p$ can not have p-rank one. Hence either $\tilde{A}^p$ is ordinary or it has p-rank 0. Q.E.D.

COROLLARY 2. *Let A be a two-dimensional abelian variety as in Corollary 1. A is simple (even absolutely simple) and it is defined over a number field K. Then for any finite p of K, $\tilde{A}^p$ is not simple.*

PROOF. By Corollary 1, either $\tilde{A}^p$ is ordinary or it has p-rank zero. The following idea of the proof is due to Deligne-Rapoport [2]. $\theta = \mathrm{End}_K(A)$ is an order in the given indefinite quaternion division algebra H. We have an embedding of θ in $\mathrm{End}_k(\tilde{A}^p)$ where k is the residue field at the prime p. Hence θ can be considered as a ring of operators of $\tilde{A}^p$. Let u be a nontrivial idempotent of θ (note that such an idempotent exists, since θ is a nonnilpotent subring of H). Then $e = u \otimes 1$ is a nontrivial idempotent of $\theta \otimes \mathbf{Z}_p$. Let $T_p(\tilde{A}^p)$ denote the Barsotti-Tate group of the 2-dimensional abelian variety $\tilde{A}^p$. It has height 4. The completion of $\theta \otimes \mathbf{Z}_p$ acts on $T_p(\tilde{A}^p)$. Write $T_p(\tilde{A}^p) = eT_p(\tilde{A}^p) \oplus (1 - e)T_p(\tilde{A}^p)$. Both these components are Barsotti-Tate subgroups of $T_p(\tilde{A}^p)$, each of height 2. Corresponding to this decomposition, $\tilde{A}^p$ can be written up to isogeny as a product of elliptic curves: $\tilde{A}^p \sim E \times F$, where E = image of the endomorphism u in $\tilde{A}^p$, whose associated Barsotti-Tate group is $eT_p(\tilde{A}^p)$ and F is the connected component of the identity of the kernel of u. Since $\tilde{A}^p$ has p-rank either 0 or 2, both E and F have p-rank either 0 or 1, since p-rank is additive and isogeny invariant. This proves that the reduction $\tilde{A}^p$ of A is everywhere nonsimple. Indeed, it is isogenous either to a product of supersingular or ordinary elliptic curves. Q.E.D.

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