A NOTE ON GOOD REDUCTION OF SIMPLE ABELIAN VARIETIES

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Abstract. In this note it is shown that the reduction of a simple abelian variety of dimension $> 2$, defined over an algebraic number field, at any finite good prime need not be simple. We give an example of a two-dimensional simple abelian variety defined over an algebraic number field whose reduction at any finite prime is isogenous either to a product of ordinary elliptic curves or to a product of supersingular elliptic curves.

It is well known [7] that an abelian variety $A$ defined over an algebraic number field $K$ has good reduction almost everywhere. In other words, if $O$ is the ring of integers in $K$, $\Sigma$ the set of all finite primes of $K$, and $S$ a finite subset of $\Sigma$, then for every prime $p$ in $\Sigma - S$, there exists an abelian scheme $X$ over $O_p$ such that $X \times_{O_p} K = A$. Serre and Tate [6] have also proved that if $A$ has sufficiently many complex multiplications, then it has potentially good reduction everywhere. It is clear that if the reduction $\tilde{A}$ of an abelian variety $A$ is simple, then $A$ itself is simple, because $\text{End}_0(A) \subseteq \text{End}_0(\tilde{A})$ and the latter is a division algebra over $\mathbb{Q}$. One can ask whether the converse is true. More precisely, if $A$ is simple, then is $\tilde{A}$ also simple? In this note we shall see that this need not be the case.

Notations and terminology. We employ the following notation:
- $\mathbb{Z}$ = the ring of rational integers; $\mathbb{Q}$ = the field of rationals;
- $\mathbb{Z}_p$ = the ring of $p$-adic integers; $\mathbb{Q}_p$ = the field of fractions of $\mathbb{Z}_p$;
- $\mathbb{Q}^*_p$ = the algebraic closure of $\mathbb{Q}_p$; $\mathbb{C}$ = the field of complex numbers.

The principal reference for abelian varieties is [4]. If $A$ is an abelian variety defined over a field $K$, let $\text{End}_0(A) = \text{End}_K(A) \otimes \mathbb{Q}$. If the characteristic of $K$ is $p > 0$, let $T_p(A)$ denote the Barsotti-Tate group ($p$-divisible group in the terminology of Serre and Tate) associated to $A$. It has height $2g$, where $g$ is the dimension of $A$. $T_p(A)_{\text{red}}$ = the étale part of $T_p(A)$. Let $M$ denote the Dieudonné module of $T_p(A)$. If $K$ is a perfect field, then $M$ is a module over $W(K)[F, V]$, where $W(K)$ is the ring of infinite Witt vectors over $K$ and $F, V$...
are indeterminates such that (i) $FV = VF = p$ and (ii) $F\alpha = \alpha F$, $\alpha V = V\alpha$ for $\alpha \in W(K)$; here $\sigma$ is the unique automorphism of $W(K)$ inducing the map $x \to x^p$ in $K$. The module $M$ is free of rank $2g$ over $W(K)$. For details on Dieudonné modules, see [5]. The integer $p(A)$ will denote the $p$-rank of an abelian variety $A$.

**Theorem.** Let $A$ be an abelian variety of dimension $\geq 2$, defined over a field $k$ of positive characteristic $p$. Let $p(A) = 1$. Then $\text{End}^0(A)$ will never contain a simple subalgebra which is not a field.

**Proof.** Suppose that $\text{End}^0(A)$ contains a simple subalgebra $L$ such that $1_A = 1_L$. Let $C$ be the center of $L$. Then from the general theory of simple algebras, we have $[L:C] = d^2$ and $[C:Q] = e$ for some natural numbers $d$ and $e$. Let $\rho$ be the representation of $L \otimes Q_p$ on the extended Dieudonné module $M \otimes Q_p$ of $T_p(A)$. This representation is induced from the representation of $\text{End}^0(A)$. It has degree $2g$ over $W(k) \otimes Q_p$. The representation $\rho$ splits into three parts corresponding to the splitting of $M$ into $M^\text{et} \oplus M^\text{loc} \oplus M^\text{loc,loc}$, where $M^\text{et}$ is the étale part of $M$ whose Cartier dual is also étale, etc. Write $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$. The first representation $\rho_1$ is the representation over $Q_p$ of $L \otimes Q_p$ on the $p$-adic Tate module $T_p(A)_{\text{red}} \otimes Q_p$. Since $p(A) = 1$, $T_p(A) \otimes Q_p$ is isomorphic to $Q_p$. Decompose $L \otimes Q_p$ into a product \( \prod M_p(Q_p), \) where $M_p(Q_p)$ is the algebra of matrices of degree $d$ over $Q_p$. Since the identity of $L$ is represented by the identity matrix, the representation $\rho_1$ does not contain the zero representation. Consequently $d = 1$; that is $L$ is a field. Q.E.D.

**Corollary 1.** Let $M_2$ be the one-dimensional moduli scheme of two-dimensional abelian varieties defined over $C$ whose endomorphism ring is an order in an indefinite quaternion division algebra $H$ over $Q$. Then the fiber at any closed point is a simple abelian variety over an algebraic number field $K$ whose reduction at every finite prime of $K$ is either ordinary or has $p$-rank 0.

**Remark.** The abelian varieties corresponding to closed points as in Corollary 1 are called *false elliptic curves* by J.-P. Serre for the reason that they behave like elliptic curves when we reduce at a finite prime. Recall that an elliptic curve in positive characteristic is either ordinary or supersingular, i.e. the $p$-rank is either one or zero.

**Proof of Corollary 1.** As Shimura has observed in [8], $M_2$ has a canonical nonsingular model defined over $Q$ which we again denote by $M_2$. Let $A$ be a two-dimensional abelian variety corresponding to any closed point of $M_2$. $A$ is defined over a number field, say, $K$. It has potential good reduction at any finite prime of $K$. For example, see [3]. By extending $K$ to a finite extension if necessary, we can assume that $A$ has good reduction everywhere. For any finite prime $p$ of $K$, let $A^p$ be the reduction of $A$ at $p$. $A^p$ is defined over a finite field of characteristic $p$. Since $\text{End}^0(A) \subseteq \text{End}^0(A^p)$, $\text{End}^0(A^p)$ contains an indefinite quaternion division algebra and so by the
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Theorem, \( \tilde{A}^p \) can not have \( p \)-rank one. Hence either \( \tilde{A}^p \) is ordinary or it has \( p \)-rank 0. Q.E.D.

**Corollary 2.** Let \( A \) be a two-dimensional abelian variety as in Corollary 1. \( A \) is simple (even absolutely simple) and it is defined over a number field \( K \). Then for any finite \( p \) of \( K \), \( A^p \) is not simple.

**Proof.** By Corollary 1, either \( \tilde{A}^p \) is ordinary or it has \( p \)-rank zero. The following idea of the proof is due to Deligne-Rapoport [2]. \( \theta = \text{End}_K(A) \) is an order in the given indefinite quaternion division algebra \( H \). We have an embedding of \( \theta \) in \( \text{End}_k(\tilde{A}^p) \) where \( k \) is the residue field at the prime \( p \). Hence \( \theta \) can be considered as a ring of operators of \( \tilde{A}^p \). Let \( u \) be a nontrivial idempotent of \( \theta \) (note that such an idempotent exists, since \( \theta \) is a non-nilpotent subring of \( H \)). Then \( e = u \otimes 1 \) is a nontrivial idempotent of \( \theta \otimes \mathbb{Z}_p \). Let \( T_p(\tilde{A}^p) \) denote the Barsotti-Tate group of the 2-dimensional abelian variety \( \tilde{A}^p \). It has height 4. The completion of \( \theta \otimes \mathbb{Z}_p \) acts on \( T_p(\tilde{A}^p) \). Write \( T_p(\tilde{A}^p) = eT_p(\tilde{A}^p) \oplus (1 - e)T_p(\tilde{A}^p) \). Both these components are Barsotti-Tate subgroups of \( T_p(\tilde{A}^p) \), each of height 2. Corresponding to this decomposition, \( \tilde{A}^p \) can be written up to isogeny as a product of elliptic curves: \( \tilde{A}^p \sim E \times F \), where \( E = \text{image of the endomorphism } u \text{ in } \tilde{A}^p \), whose associated Barsotti-Tate group is \( eT_p(\tilde{A}^p) \) and \( F \) is the connected component of the identity of the kernel of \( u \). Since \( \tilde{A}^p \) has \( p \)-rank either 0 or 2, both \( E \) and \( F \) have \( p \)-rank either 0 or 1, since \( p \)-rank is additive and isogeny invariant. This proves that the reduction \( \tilde{A}^p \) of \( A \) is everywhere nonsimple. Indeed, it is isogenous either to a product of supersingular or ordinary elliptic curves. Q.E.D.

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**References**


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