EXTREMAL INTERPOLATORY FUNCTIONS IN $H^\infty$

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ABSTRACT. Let a Blaschke sequence $(z_n)$ and a bounded sequence $(w_n)$ be given. If we can find an $f$ in $H^\infty$ such that $f(z_n) = w_n$ we may assume that $\|f\|$ is minimal. Such an $f$ need not be unique, but a sufficient condition for uniqueness is given. Properties of $f$ in the case of uniqueness are studied.

Introduction. Let $H^\infty$ be the Banach space of bounded analytic functions on $D = \{z: |z| < 1\}$, and let $T = \{z: |z| = 1\}$. We call a sequence $(z_n)$ in $D$ interpolating if for every bounded sequence $(w_n)$ we can find an $f$ in $H^\infty$ such that $f(z_n) = w_n$ for all $n$. A theorem of Carleson states that $(z_n)$ is interpolating if and only if $\inf_n \prod_{k \neq n} |(z_k - z_n)/(1 - z_k z_n)| = \delta > 0$. Such a sequence is also called uniformly separated. In particular interpolating sequences have to be Blaschke sequences, that is $\sum (1 - |z_n|)$ is finite.

Given a Blaschke sequence $(z_n)$ and a bounded sequence $(w_n)$. If we can find an $f$ in $H^\infty$ such that $f(z_n) = w_n$ for all $n$, we may assume that $\|f\|$ is minimal. Such a function is called extremal. Necessary and sufficient conditions for uniqueness of an extremal interpolating function have been given by Denjoy and Nevanlinna, but their conditions are very implicit. See [4].

Theorem 1 below gives a necessary condition for uniqueness. Theorem 2 below gives a sufficient condition for uniqueness and analytic continuation of the extremal function is studied. Akutowicz and Carleson [1] have also studied analytic continuation of extremal interpolatory functions. Theorem 3 studies the unique function of Theorem 2 in a special case.

The problem is a special case of a more general question: Given $F \in L^\infty$, when does the coset $F + H^\infty$ in $L^\infty / H^\infty$ have a unique element of least norm? See [2].

**Theorem 1.** Let $\alpha$ be an accumulation point of the Blaschke sequence $(z_n)$. If $f$ is continuous at $\alpha$, and if $\|f\| = |f(\alpha)| = 1$, and if $f$ is not an extreme point of the unit ball of $H^\infty$, then uniqueness fails.

**Proof.** If $f$ is not an extreme point of the unit ball, then

$$\int_{-\pi}^\pi \log(1 - |f(e^{i\theta})|) \, d\theta > -\infty.$$ 

Let $B(z) = \prod_{n=1}^\infty |z_n| / z_n \cdot (z_n - z)/(1 - \bar{z}_n z)$ and

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We see that $|f(e^{i\theta})| + |B(e^{i\theta}) \cdot h(e^{i\theta})| < 1$ hence $f + Bh$ is another interpolating function of minimal norm.

We will now prove uniqueness in a special case:

**Theorem 2.** Let $\{z_n\}$ be a uniformly separated sequence in $D$ and assume $w_n \to 0$. Then there exists a unique $f$ in $H^\infty$ of minimal norm such that $f(z_n) = w_n$ for all $n$. This function is a complex constant times an inner function and has analytic continuation across $T \setminus \{z_n\}$.

**Proof.** Let $B(z)$ be as above

$$kB(z) = B(z) \cdot \frac{1 - \bar{z}_k z}{z_k - z},$$

$$kB_N(z) = B_N(z) \cdot \frac{1 - \bar{z}_k z}{z_k - z}, \quad k < N.$$

Let $\delta_{N,n} = nB_N(z_n)$, $\delta_n = nB(z_n)$. We have $|\delta_{N,n}| > |\delta_n| > \delta > 0$ for some $\delta$ since $\{z_n\}$ is uniformly separated. It is not hard to show that $dz/B_N(z) \to dz/B(z)$ in the $w^*$-topology of the measures on $T$. Using that the polynomials are dense in $H^1$, we obtain that $\int_T (h(z)/B_N(z)) \, dz \to \int_T (h(z)/B(z)) \, dz$ for all $h \in H^1$. Since $\{z_n\}$ is uniformly separated, there exists an $f \in H^\infty$ of minimal norm such that $f(z_n) = w_n$ for all $n$. It is well known that

$$\|f\| = \sup_{h \in H^1, \|h\| < 1} \left| \frac{1}{2\pi i} \int_T \frac{f(z)}{B_N(z)} h(z) \, dz \right|$$

and that $f$ is unique if this sup is attained. See [5, p. 132]. Calculation gives

$$\frac{1}{2\pi i} \int_T \frac{f(z)}{B_N(z)} h(z) \, dz = \sum_{n=1}^N \frac{w_n}{\delta_{N,n}} h(z_n)(1 - |z_n|^2).$$

There exists a constant $K$ such that $\Sigma |h(z_n)|(1 - |z_n|^2) < K \cdot \|h\|$ for all $h \in H^1$. A proof is given in [5, Chapter 9]. Since $\delta_{N,n} \to \delta_n$ when $N \to \infty$ and $|\delta_{N,n}| > \delta > 0$, we have

$$\frac{1}{2\pi i} \int_T \frac{f(z)}{B(z)} h(z) \, dz = \sum_{n=1}^\infty \frac{w_n}{\delta_n} h(z_n)(1 - |z_n|^2) \quad \text{for all } h \in H^1.$$

Let $\varepsilon > 0$ be given. Then there exists an integer $N$ such that

\[\sum_{n=N}^\infty \left| \frac{w_n}{\delta_n} h(z_n)(1 - |z_n|^2) \right| < \varepsilon \cdot \|h\|\]  \[\text{(\dagger)}\]

This is true since $w_n \to 0$. Choose a sequence $h_k \in H^1$, $\|h_k\| = 1$ such that
\[
\sum_{n=1}^{\infty} \frac{(w_n/\delta_n)h_k(z_n)(1 - |z_n|^2)}{2\pi} \to \|f\| \quad \text{when } k \to \infty.
\]

We may assume that \( h_k \to h \in H^1 \) uniformly on compacts. \( \|h\| < 1. \) The relation (*) gives that \( \|f\| = \sum_{n=1}^{\infty} \frac{(w_n/\delta_n)h(z_n)(1 - |z_n|^2)}{2\pi} \). Thus the sup is attained and \( f \) is unique. For the extremal function \( h \) we have

\[
\|f\| = \frac{1}{2\pi} \int_{\Gamma} \frac{f(z)}{|B(z)|} h(z) \, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{|B(e^{i\theta})|} e^{i\theta} h(e^{i\theta}) \, d\theta.
\]

Since \( (1/2\pi)\int_{-\pi}^{\pi} |h(e^{i\theta})| \, d\theta = 1 \) and \( h(e^{i\theta}) \neq 0 \) a.e. we have \( |f(e^{i\theta})| = \|f\| \) a.e. This shows that \( f \) is a complex constant times an inner function. We also have that \( (f(e^{i\theta})/B(e^{i\theta})) e^{i\theta} h(e^{i\theta}) \) is real a.e. Let \( \Gamma = T \setminus \{z_n\} \). Every point \( x \in \Gamma \) has a neighbourhood \( O_x \) in \( D \) where \( |(f(z)/B(z))h(z) \cdot z| \) has a harmonic majorant (i.e. lies locally in \( H^1 \)). This is true since \( h(z) \in H^1 \). This shows that \( (f(z)/B(z))z \cdot h(z) \) and hence also \( f(z)h(z) \) has analytic continuation across \( \Gamma \).

Because \( fh \) is analytic across \( \Gamma \) and \( f \) is inner, \( f \) is analytic across \( \Gamma \). See pp. 68–70 of [7]. This proves Theorem 2.

An interesting question is now whether the unique inner function of Theorem 2 is a Blaschke product or not. We will prove that the answer is affirmative under an additional condition:

**Theorem 3.** Assume that \( \{z_n\} \) is uniformly separated and that \( z_n \to 1 \) non tangentially. Then the unique function \( f \) of Theorem 2 is a constant times a Blaschke product.

**Proof.** We know that \( f(z) = \lambda S(z)B(z) \) and that \( f \) has analytic continuation across \( T \setminus \{1\} \). This shows that \( S(z) = \exp(((z + 1)/(z - 1)))^{\gamma} \) for some \( \gamma > 0 \). We have to prove that \( \gamma = 0 \).

**Case I.** Assume \( w_n \) is not \( O(1 - z_n)^4 \). In this case \( \gamma = 0 \) because it is easy to prove that \( S(z) = O(1 - z)^4 \) when \( \gamma > 0 \) and \( z \) lies in a Stoltz angle.

**Case II.** Assume \( w_n = O(1 - z_n)^4 \). The sequence \( \{z_n\} \) lies in a Stoltz angle bounded by rays \( L_1 \) and \( L_2 \). Choose a larger Stoltz angle bounded by \( L_1' \) and \( L_2' \) that contains the first Stoltz angle. The new angle divides \( D \) into three parts, one large part and two segments \( A_1 \) and \( A_2 \). Let

\[
\psi(a, b) = |(a - b)/(1 - \bar{a}b)|
\]

be the pseudohyperbolic metric on \( D \). An easy calculation shows that \( \psi\{z_n\}, A_i > 0, i = 1, 2. \)

Let \( B_1(z) = \prod_{n=1}^{\infty} |z_n|/(z_n \cdot (z_n - 1)/(1 - \bar{z}n)). \) A well-known result [8] now shows that \( |B_1(z)| > \delta > 0 \) for \( z \in A_i \). Let \( \Omega' = \{z: |B(z)| < \delta^{-1}, |z| < 2\} \) and let \( \Omega \) be the maximal star shaped subset of \( \Omega' \) w.r.t. 0.

The proof of Lemma 1 in [6] shows that \( \{z_n\} \) is interpolating for \( H^\infty(\Omega) \). Let \( \alpha_n = w_n(1 - z_n)^{-4} \). Since the numbers \( \alpha_n \) are bounded, we can find a function \( g \in H^\infty(\Omega) \) such that \( g(z_n) = \alpha_n \) for all \( n \). Let \( h(z) = (1 - z)\alpha g(z), h(z_n) = w_n \). We want to prove that \( (h(z)/B_1(z))|T \in C^1(T) \). Assume this is
proved. Then \( \inf_{f \in H^\infty} \| h/B_1 + j \| = \| h/B_1 + k \| \) where \( k \in A(D) \), the disc algebra. This is proved by Carleson and Jacobs in [3]. Since
\[
\inf_{f \in H^\infty} \left\| \frac{h}{B_1} + j \right\| = \inf_{f \in H^\infty} \| h + B_1j \|,
\]
we have that \( h + B_1k = f = \lambda BS \) by Theorem 2. Since \( h \in A(D) \) and \( h(1) = 0 \), we must have \( |k(1)| = |\lambda| \neq 0 \). \( h(z) \) is \( O(1 - z) \) in the largest Stoltz angle, but \( B_1z(k)(z) \) is not since \( k(1) \neq 0 \) and \( |B_1(z)| > \delta > 0 \) for \( z \in A_1 \). This shows that \( h + B_1k \) is not \( O(1 - z) \) in the Stoltz angle, hence \( S(z) \equiv 1 \).

It remains to prove that \( p(z) = h(z)/B_1(z) \) is in \( C^1(T) \). It is clear that \( p'(e^{i\theta}) \) exists and is continuous for \( \theta \neq 0 \), and that \( p'(1) = 0 \). The derivative is taken w.r.t. \( \theta \). It is sufficient to prove that \( \lim_{z \to 1; |z| = 1} p'(z) = 0 \) where the derivative is taken w.r.t. \( z \).

For \( |z| = 1 \) and \( z \) near 1 let \( r(z) \) be the distance from \( z \) to \((L_1 \cup L_2)\). It is easy to see that \( K_1 |1 - z| > r(z) > K_2 |1 - z| \) for constants \( K_1 \) and \( K_2 \) independent of \( z \), and that the disc \( D_z = \{ w: |z - w| < r(z) \} \) is contained in \( \Omega \). Cauchy’s formula gives:
\[
P'(z) = \frac{1}{2\pi i} \int_{\partial D_z} \frac{h(w)}{B_1(w)(z - w)^2} \, dw = \frac{1}{2\pi i} \int_{\partial D_z} \frac{(1 - w) \overline{g(w)}}{B_1(w)(z - w)^2} \, dw.
\]

Hence
\[
|p'(z)| < \frac{\|g\|_\infty}{\delta} \int_{\partial D_z} \frac{|1 - w|^4}{r(z)^2} \, dw < K \frac{1}{r(z)^2} 2\pi r(z)(r(z) + |1 - z|)^4
\]
\[
< K' \frac{1}{K_1 |1 - z|} (K_2 |1 - z| + |1 - z|)^4 \to 0 \quad z \to 1; |z| = 1.
\]

This completes the proof.

If \( \{ w_n \} \) is a constant sequence, there is of course a unique interpolating function of minimal norm. From this fact and Theorem 2 one is led to the conjecture that if \( \{ w_n \} \) is “smooth enough” one has uniqueness. This is not true in the following sense:

**Theorem 4.** If the bounded sequence \( \{ w_n \} \) satisfies \( |w_k| < \sup_{n} |w_n| \) for all \( k \), there exists a uniformly separated sequence \( \{ z_n \} \) such that there are more than one function of minimal norm interpolating \( \{ w_n \} \) in \( \{ z_n \} \).

**Proof.** We may assume \( \sup_{n} |w_n| = 1 \). Choose ball \( f \in H^\infty \) such that \( f \) is not extreme and such that \( f(0_0) = D \) for every set \( 0_0 \) of the form \( \{ z: |z| < 1, |z - 1| < t \} \). Choose \( z_1 \in D \) such that \( f(z_1) = w_1 \). Assume \( z_1, \ldots, z_n \) have been chosen. Take \( z_{n+1} \in D \) such that \( f(z_{n+1}) = w_{n+1} \) and
\[
1 - |z_{n+1}| < \frac{1}{2} (1 - |z_n|).
\]

\( \{ z_n \} \) is uniformly separated by [7, p. 203]. \( f \) is of minimal norm but not unique by Theorem 1.

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REFERENCES


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