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EXTREMAL INTERPOLATORY FUNCTIONS IN H^∞

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ABSTRACT. Let a Blaschke sequence $\{z_n\}$ and a bounded sequence $\{w_n\}$ be given. If we can find an f in H^∞ such that $f(z_n) = w_n$ we may assume that $\|f\|$ is minimal. Such an f need not be unique, but a sufficient condition for uniqueness is given. Properties of f in the case of uniqueness are studied.

Introduction. Let H^∞ be the Banach space of bounded analytic functions on $D = \{z: |z| < 1\}$, and let $T = \{z: |z| = 1\}$. We call a sequence $\{z_n\}$ in D interpolating if for every bounded sequence $\{w_n\}$ we can find an f in H^∞ such that $f(z_n) = w_n$ for all n. A theorem of Carleson states that $\{z_n\}$ is interpolating if and only if $\inf_n \prod_{k \neq n} |(z_k - z_n)/(1 - z_k \bar{z}_n)| = \delta > 0$. Such a sequence is also called uniformly separated. In particular interpolating sequences have to be Blaschke sequences, that is $\Sigma(1 - |z_n|) < \infty$.

Given a Blaschke sequence $\{z_n\}$ and a bounded sequence $\{w_n\}$. If we can find an f in H^∞ such that $f(z_n) = w_n$ for all n, we may assume that $\|f\|$ is minimal. Such a function is called extremal. Necessary and sufficient conditions for uniqueness of an extremal interpolating function have been given by Denjoy and Nevanlinna, but their conditions are very implicit. See [4].

Theorem 1 below gives a necessary condition for uniqueness. Theorem 2 below gives a sufficient condition for uniqueness and analytic continuation of the extremal function is studied. Akutowicz and Carleson [1] have also studied analytic continuation of extremal interpolatory functions. Theorem 3 studies the unique function of Theorem 2 in a special case.

The problem is a special case of a more general question: Given $F \in L^\infty$, when does the coset $F + H^\infty$ in L^∞/H^∞ have a unique element of least norm? See [2].

THEOREM 1. *Let α be an accumulation point of the Blaschke sequence $\{z_n\}$. If f is continuous at α, and if $\|f\| = |f(\alpha)| = 1$, and if f is not an extreme point of the unit ball of H^∞, then uniqueness fails.*

PROOF. If f is not an extreme point of the unit ball, then

$$\int_{-\pi}^{\pi} \log\left(1 - |f(e^{i\theta})|\right) d\theta > -\infty.$$

Let $B(z) = \prod_{n=1}^{\infty} |z_n|/z_n \cdot (z_n - z)/(1 - \bar{z}_n z)$ and

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$$h(z) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\left(1 - |f(e^{i\theta})|\right) d\theta \right\}.$$

We see that $|f(e^{i\theta})| + |B(e^{i\theta}) \cdot h(e^{i\theta})| \leq 1$ hence $f + Bh$ is another interpolating function of minimal norm.

We will now prove uniqueness in a special case:

THEOREM 2. *Let $\{z_n\}$ be a uniformly separated sequence in D and assume $w_n \to 0$. Then there exists a unique f in H^∞ of minimal norm such that $f(z_n) = w_n$ for all n. This function is a complex constant times an inner function and has analytic continuation across $T \setminus \overline{\{z_n\}}$.*

PROOF. Let $B(z)$ be as above

$$B_N(z) = \prod_{n=1}^{N} \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z},$$

$$_kB(z) = B(z) \cdot \frac{1 - \bar{z}_k z}{z_k - z},$$

$$_kB_N(z) = B_N(z) \cdot \frac{z_k}{|z_k|} \cdot \frac{1 - \bar{z}_k z}{z_k - z}, \qquad k \leq N.$$

Let $\delta_{N,n} = {}_nB_N(z_n)$, $\delta_n = {}_nB(z_n)$. We have $|\delta_{N,n}| \geq |\delta_n| > \delta > 0$ for some δ since $\{z_n\}$ is uniformly separated. It is not hard to show that $dz/B_N(z) \to dz/B(z)$ in the w^*-topology of the measures on T. Using that the polynomials are dense in H^1, we obtain that $\int_T (h(z)/B_N(z))\, dz \to \int_T (h(z)/B(z))\, dz$ for all $h \in H^1$. Since $\{z_n\}$ is uniformly separated, there exists an $f \in H^\infty$ of minimal norm such that $f(z_n) = w_n$ for all n. It is well known that

$$\|f\| = \sup_{h \in H^1 \|h\| \leq 1} \left| \frac{1}{2\pi i} \int_T \frac{f(z)}{B_N(z)} h(z)\, dz \right|$$

and that f is unique if this sup is attained. See [5, p. 132]. Calculation gives

$$\frac{1}{2\pi i} \int_T \frac{f(z)}{B_N(z)} h(z)\, dz = \sum_{n=1}^{N} \frac{w_n}{\delta_{N,n}} h(z_n)\left(1 - |z_n|^2\right).$$

There exists a constant K such that $\Sigma |h(z_n)|(1 - |z_n|^2) \leq K \cdot \|h\|$ for all $h \in H^1$. A proof is given in [5, Chapter 9]. Since $\delta_{N,n} \to \delta_n$ when $N \to \infty$ and $|\delta_{N,n}| > \delta > 0$, we have

$$\frac{1}{2\pi i} \int_T \frac{f(z)}{B(z)} h(z)\, dz = \sum_{n=1}^{\infty} \frac{w_n}{\delta_n} h(z_n)\left(1 - |z_n|^2\right) \quad \text{for all } h \in H^1.$$

Let $\varepsilon > 0$ be given. Then there exists an integer N such that

$$(*) \qquad \sum_{n=N}^{\infty} \left| \frac{w_n}{\delta_n} h(z_n)\left(1 - |z_n|^2\right) \right| < \varepsilon \cdot \|h\|.$$

This is true since $w_n \to 0$. Choose a sequence $h_k \in H^1$, $\|h_k\| = 1$ such that

$$\sum_{n=1}^{\infty} (w_n/\delta_n) h_k(z_n)\left(1 - |z_n|^2\right) \to \|f\| \quad \text{when } k \to \infty.$$

We may assume that $h_k \to h \in H^1$ uniformly on compacts. $\|h\| \le 1$. The relation (∗) gives that $\|f\| = \sum_{n=1}^{\infty}(w_n/\delta_n)h(z_n)(1 - |z_n|^2)$. Thus the sup is attained and f is unique. For the extremal function h we have

$$\|f\| = \frac{1}{2\pi i}\int_T \frac{f(z)}{B(z)} h(z)\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{B(e^{i\theta})} e^{i\theta} h(e^{i\theta})\, d\theta.$$

Since $(1/2\pi)\int_{-\pi}^{\pi}|h(e^{i\theta})|\, d\theta = 1$ and $h(e^{i\theta}) \neq 0$ a.e. we have $|f(e^{i\theta})| = \|f\|$ a.e. This shows that f is a complex constant times an inner function. We also have that $(f(e^{i\theta})/B(e^{i\theta}))e^{i\theta}h(e^{i\theta})$ is real a.e. Let $\Gamma = T \setminus \overline{\{z_n\}}$. Every point $x \in \Gamma$ has a neighbourhood O_x in $\bar{D}$ where $|(f(z)/B(z))h(z) \cdot z|$ has a harmonic majorant (i.e. lies locally in H^1). This is true since $h(z) \in H^1$. This shows that $(f(z)/B(z))z \cdot h(z)$ and hence also $f(z)h(z)$ has analytic continuation accross Γ.

Because fh is analytic across Γ and f is inner, f is analytic across Γ. See pp. 68–70 of [7]. This proves Theorem 2.

An interesting question is now whether the unique inner function of Theorem 2 is a Blaschke product or not. We will prove that the answer is affirmative under an additional condition:

Theorem 3. *Assume that $\{z_n\}$ is uniformly separated and that $z_n \to 1$ nontangentially. Then the unique function f of Theorem 2 is a constant times a Blaschke product.*

Proof. We know that $f(z) = \lambda S(z)B(z)$ and that f has analytic continuation across $T \setminus \{1\}$. This shows that $S(z) = \exp\{((z + 1)/(z - 1))\gamma\}$ for some $\gamma \ge 0$. We have to prove that $\gamma = 0$.

Case I. Assume w_n is not $O(1 - z_n)^4$. In this case $\gamma = 0$ because it is easy to prove that $S(z) = O(1 - z)^4$ when $\gamma > 0$ and z lies in a Stoltz angle.

Case II. Assume $w_n = O(1 - z_n)^4$. The sequence $\{z_n\}$ lies in a Stoltz angle bounded by rays L_1 and L_2. Choose a larger Stoltz angle bounded by L_1' and L_2' that contains the first Stoltz angle. The new angle divides D into three parts, one large part and two segments A_1 and A_2. Let

$$\psi(a, b) = |(a - b)/(1 - \bar{a}b)|$$

be the pseudohyperbolic metric on D. An easy calculation shows that $\psi(\{z_n\}, A_i) > 0, i = 1, 2$.

Let $B_1(z) = \prod_{n=1}^{\infty}|z_n|/z_n \cdot (z_n - z)/(1 - \bar{z}_n z)$. A well-known result [8] now shows that $|B_1(z)| > \delta > 0$ for $z \in A_i$. Let $\Omega' = \{z\colon |B(z)| < \delta^{-1}, |z| < 2\}$ and let Ω be the maximal star shaped subset of Ω' w.r.t. 0.

The proof of Lemma 1 in [6] shows that $\{z_n\}$ is interpolating for $H^\infty(\Omega)$. Let $\alpha_n = w_n(1 - z_n)^{-4}$. Since the numbers α_n are bounded, we can find a function $g \in H^\infty(\Omega)$ such that $g(z_n) = \alpha_n$ for all n. Let $h(z) = (1 - z)^4 g(z)$, $h(z_n) = w_n$. We want to prove that $(h(z)/B_1(z))|T \in C^1(T)$. Assume this is proved. Then $\inf_{j \in H^\infty} \|h/B_1 + j\| = \|h/B_1 + k\|$ where $k \in A(D)$, the disc algebra. This is proved by Carleson and Jacobs in [3]. Since

$$\inf_{j \in H^\infty} \left\| \frac{h}{B_1} + j \right\| = \inf_{j \in H^\infty} \|h + B_1 j\|,$$

we have that $h + B_1 k = f = \lambda BS$ by Theorem 2. Since $h \in A(D)$ and $h(1) = 0$, we must have $|k(1)| = |\lambda| \neq 0$. $h(z)$ is $O(1 - z)$ in the largest Stoltz angle, but $B_1(z)k(z)$ is not since $k(1) \neq 0$ and $|B_1(z)| > \delta > 0$ for $z \in A_i$. This shows that $h + B_1 k$ is not $O(1 - z)$ in the Stoltz angle, hence $S(z) \equiv 1$.

It remains to prove that $p(z) = h(z)/B_1(z)$ is in $C^1(T)$. It is clear that $p'(e^{i\theta})$ exists and is continuous for $\theta \neq 0$, and that $p'(1) = 0$. The derivative is taken w.r.t. θ. It is sufficient to prove that $\lim_{z \to 1; |z| = 1} p'(z) = 0$ where the derivative is taken w.r.t. z.

For $|z| = 1$ and z near 1 let $r(z)$ be the distance from z to $(L_1' \cup L_2')$. It is easy to see that $K_2|1 - z| > r(z) > K_1|1 - z|$ for constants K_1 and K_2 independent of z, and that the disc $D_z = \{w: |z - w| < r(z)\}$ is contained in Ω. Cauchy's formula gives:

$$p'(z) = \frac{1}{2\pi i} \int_{\partial D_z} \frac{h(w)}{B_1(w)(z - w)^2}\, dw = \frac{1}{2\pi i} \int_{\partial D_z} \frac{(1 - w)^4 g(w)}{B_1(w)(z - w)^2}\, dw.$$

Hence

$$|p'(z)| \leq \frac{\|g\|_\Omega}{\delta} \int_{\partial D_z} \frac{|1 - w|^4}{r(z)^2} |dw| < K \frac{1}{r(z)^2} 2\pi r(z)(r(z) + |1 - z|)^4$$

$$< K' \frac{1}{K_1|1 - z|} (K_2|1 - z| + |1 - z|)^4 \underset{z \to 1;\, |z| = 1}{\longrightarrow} 0.$$

This completes the proof.

If $\{w_n\}$ is a constant sequence, there is of course a unique interpolating function of minimal norm. From this fact and Theorem 2 one is led to the conjecture that if $\{w_n\}$ is "smooth enough" one has uniqueness. This is not true in the following sense:

THEOREM 4. *If the bounded sequence $\{w_n\}$ satisfies $|w_k| < \sup_n |w_n|$ for all k, there exists a uniformly separated sequence $\{z_n\}$ such that there are more than one function of minimal norm interpolating $\{w_n\}$ in $\{z_n\}$.*

PROOF. We may assume $\sup_n |w_n| = 1$. Choose ball $f \in H^\infty$ such that f is not extreme and such that $f(0_t) = D$ for every set 0_t of the form $\{z: |z| < 1, |z - 1| < t\}$. Choose $z_1 \in D$ such that $f(z_1) = w_1$. Assume $z_1, \ldots, z_n$ have been chosen. Take $z_{n+1} \in D$ such that $f(z_{n+1}) = w_{n+1}$ and

$$1 - |z_{n+1}| < \tfrac{1}{2}(1 - |z_n|).$$

$\{z_n\}$ is uniformly separated by [7, p. 203]. f is of minimal norm but not unique by Theorem 1.

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