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SOME INVARIANT PROPERTIES ON SUMMABILITY DOMAINS

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ABSTRACT. Let A be an infinite matrix. Each $f \in c_A'$ has a representation $f(x) = \alpha \lim_A x + t(Ax) + rx$. The purpose of this short article is to answer the following problems raised by Wilansky. 1. Does α invariantly unique imply $\alpha^\perp$ invariant? 2. Does A not-replaceable imply $\alpha^\perp$ invariant? 3. Could a function $f \in c_A'$ with α uniquely zero have a matrix representation? 4. Is the set of test functions invariant?

We assume that A is a fixed matrix with convergent columns, i.e. $c_A \supset \varphi$, the finite sequences. Every $f \in c_A'$ has a representation

$$f(x) = \alpha \lim_A x + t(Ax) + rx \tag{1}$$

where $t \in l$, $r \in c_A^\beta$, $t(Ax) = \sum_n t_n (Ax)_n$, $rx = \sum_k r_k x_k$.

The representation (1) is far from unique for f and we say α is unique for A if all representations for f have the same α. It is easy to see that α is unique for one f iff α is unique for all f.

α is said to be invariantly unique of α is unique for every B with $c_A = c_B$. If α is invariantly unique, and B is any matrix with $c_A = c_B$, and $f \in c_A'$, we write $\alpha_A(f)$, $\alpha_B(f)$ for the values of α when f is expressed in the form (1) with respect to A or B. Put $\alpha_A^\perp = \{f \in c_A';\ \alpha_A(f) = 0\}$ and similarly for $\alpha_B^\perp$. If $\alpha_A^\perp = \alpha_B^\perp$ for every B with $c_A = c_B$, we say $\alpha^\perp$ is invariant.

The following problems were raised by Wilansky in [1].

1. Does α invariantly unique imply $\alpha^\perp$ invariant?

2. Does A not-replaceable imply $\alpha^\perp$ invariant? (Here we assume A is conversative.)

3. Could a function $f \in c_A'$ with α uniquely zero have a matrix representation? I.e. there is a matrix B with $c_A = c_B$, $\lim_B = f$.

It is known that if f has a representation (1) with $\alpha \neq 0$, f has a matrix representation. (See [3, Satz 5.3].) We observe that *if α is not invariantly unique, there is a matrix D with $c_D = c_A$ such that f has a representation* (1) (*in D-form*) *with $\alpha \neq 0$. The above known result which we have just mentioned tells us that f has a matrix representation*. Thus it remains to consider Problem 3 in the case that α is invariantly unique.

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A function $f \in c'_A$ is called a test function if $f = 0$ on φ and $\alpha = 0$ in some representaton of f.

The next problem was raised by Wilansky in [2].

4. Is the set of test functions invariant?

All these problems can be solved by use of the following factorization theorem given in [2].

THEOREM 1. *Let A satisfy $c_A \supset \varphi$ and suppose that α is unique for A. Let $c_B \supset c_A$. Then there exist matrices C, D such that* (a) $B = CA + D$, (b) $\|C\| < \infty$, *and for all $x \in c_A$, $t \in l$, $y = Ax$ we have*: (c) $tC \in l$, (d) $(CA)x = C(Ax)$, (e) $t(Dx) = (tD)x$, (f) $t(Cy) = (tC)y$.

The next four theorems answer the above four problems respectively.

THEOREM 2. *If α is invariantly unique, then $\alpha^\perp$ is invariant.*

PROOF. Let A, B be two matrices with $c_A = c_B$ and let $f \in \alpha_B^\perp$ and so f can be written in the form $f(x) = t(Bx) + rx$. Then, by Theorem 1,

$$f(x) = t\left[(CA + D)x\right] + rx = (tC)(Ax) + \left[(tD) + r\right]x.$$

Since $tC \in l$ we have $f \in \alpha_A^\perp$. Hence $\alpha_B^\perp \subset \alpha_A^\perp$. Similarly we can prove $\alpha_A^\perp \subset \alpha_B^\perp$. Thus we conclude that $\alpha^\perp$ is invariant.

THEOREM 3. *If A is not-replaceable, then $\alpha^\perp$ is invariant.*

PROOF. Since the replaceability is an invariant property, then [1, Theorem 2.3] α is invariantly unique. Thus this theorem follows from Theorem 2.

THEOREM 4. *Let $f \in c'_B$ with $\alpha_B(f)$ uniquely zero. If α is invariantly unique, then f could not have a matrix representation.*

PROOF. Suppose there is a matrix A with $c_B = c_A$ such that $f = \lim_A$. Then, by Theorem 1,

$$\lim_A x = t(Bx) + rx = t\left[(CA + D)x\right] + rx = (tC)(Ax) + \left[tD + r\right]x.$$

But this would imply $0 = \lim_A x + t'(Ax) + r'x$ where $t' = -tC$, $r' = tD + r$. This contradicts that α is invariantly unique.

THEOREM 5. *The set of test functions is invariant.*

PROOF. Let A, B be two matrices with $c_A = c_B$. Let T_A be the set of all test functions with respect to A and similarly for T_B. If $f \in T_B$ we consider the following two cases:

CASE 1. α is not unique for A. We can write f in two different representations

$$f(x) = \alpha_1 \lim_A x + t_1(Ax) + r_1 x = \alpha_2 \lim_A x + t_2(Ax) + r_2 x.$$

Let $\lambda = \alpha_2/(\alpha_2 - \alpha_1)$. Then $f = \lambda f + (1 - \lambda) f$ expresses f with $\alpha = 0$.

CASE 2. α is unique for A. Since $f \in T_B$, f can be expressed in the form $f(x) = t(Bx) + rx$. Then, by Theorem 1,

$$f(x) = t[(CA + D)x] + rx = (tC)(Ax) + [(tD + r)]x$$

with $tC \in l$.

Thus f always has a representation (1) with $\alpha = 0$ with respect to A. Of course $f = 0$ on φ in c_A since $f \in T_B$ and $c_A = c_B$. So $f \in T_A$ and hence $T_B \subset T_A$. Similarly we can prove $T_A \subset T_B$. This completes the proof.

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